ASM Sc. J., 12, Special Issue 1, 2019 for IQRAC2018, 174-181

Z-number α-Cut For DEA Using Trapezoidal Fuzzy Numbers

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Data envelopment analysis (DEA) is a powerful tool for measuring efficiency of multiple inputs and outputs of a set of decision making units (DMUs). It is pioneered by (Farrell, 1957) but recent series of discussion started with Charnes, Cooper and Rhodes (1978). In real life, data is usually vague especially when it is characterized by linguistic information given by experts. Hence, several methods have been proposed to deal with the vagueness. Currently, the most popular method to capture the vagueness of data is the fuzzy data envelopment analysis (FDEA) which is based on α-cut. However, the limitation of this method is that the α-cut value is a crisps value given by the experts and it does not include the uncertainty of the expert judgement. In this paper, we propose the Z-number α-cut technique using the trapezoidal fuzzy numbers that includes some uncertainty information on the judgement given by the experts. A numerical example on portfolio selection in IS/IT (Information Systems/Information Technology) is presented to demonstrate the proposed method and to the efficiency score of the portfolios.
Keywords: α-cut method, Data envelopment analysis (DEA), fuzzy data envelopment analysis (FDEA), Z-number.

I. Introduction

Data envelopment analysis (DEA) is pioneered by M. J. Farrell that developed a technique on measuring a production efficiency of an organization (Farrell, 1957). Recent series of discussions on this topic started with (Charnes et al., 1978)(CCR model) that evaluate the efficiency of DMU by taking the maximum ratio of weighted outputs to weighted inputs subject to the condition that similar ratios for every DMU must be less than or equal to 1. Later (Banker et al., 1984) developed a new model, the Banker, Cooper and Rhodes (BCC) model that referred to the efficient boundary in measuring the efficiency of the DMUs. Other researchers made improvements or extensions to the CCR and BCC methods which can be classified into many streams such as the cross-efficiency, super-efficiency and many more (Adler et al., 2002).

Traditionally, all inputs and outputs values of DMUs are crisp data, but in real problems usually the data are imprecise. Several approaches have been developed to deal with fuzzy data in DEA. The first person that inserted the fuzziness into DEA model is (Sengupta, 1992). (Karsak, 2008) categorized the fuzzy theory in DEA into four groups : The tolerance approach, the α-level approach, the possibility approach and the fuzzy ranking approach. However, the α-level approach is the most popular approach in fuzzy data envelopment analysis (FDEA) model due to it simpler calculation. In this approach, FDEA model is converted into a pair of parametric programs in obtaining the lower and upper bound efficiency score. (Meada et al., 1998) used α-cut approach to obtain fuzzy interval efficiency DMUs. After a couple of years, (Kao and Liu,

2000) developed the technique of transforming the FDEA model to DEA model. The drawback of the conventional methods is that it only compare the left and right hand side of the interval. (Saati et al., 2002) introduced fuzzy CCR model that could be solved as a crisp LP model for a given α by the experts and the advantage of this method is that the variables are defined in the interval such that they satisfy the set of constraints and at the same time the objective function is maximized. Problem arises when expert usually prefers to give the α value in a fuzzy form rather than giving a certain value of it. Besides, we also concern about the uncertainty information of the α value given by the expert. As far as concerned, all of the conventional α -cut method only focused on a certain α cut value. Therefore, the conventional α-cut method is less efficient to tackle this uncertainty issue.

In 2011, Zadeh introduced the Z-number that takes into account expert's reliability on the data (Zadeh, 2011). The Z-number has two components, Z=(A, B) in estimating a variable, Y. (A) is the limitation on the values which Y can take in triangular fuzzy numbers. The second number (B) is a degree of reliability (certainty) that Y is A. This method helps experts in giving a reliable judgement as it includes the certainty value of the experts. In 2016, (Azadeh and Kokabi, 2016) integrated the Z-number approach with CCR model using triangular fuzzy numbers to handled data that are expert-based. However, according (Herrera and Herrera-Viedma, 2000) the trapezoidal fuzzy number is better suited to capture the vagueness of linguistic assessment. Therefore, (Ibrahim et al., 2018) developed Z-number CCR model using trapezoidal fuzzy numbers to overcome this problem. The formulation of this model is shown in equation 1. Both triangular and trapezoidal Z-number CCR model are then converted into fuzzy data envelopment model (FDEA) in order to linearized the model. In linearizing the FDEA model, they used a crisps α-cut value (given by the experts). As Z-number is proved can help the experts in giving their reliability towards the data, in this paper, an integration of α-cut and Z-number by using trapezoidal fuzzy numbers is proposed to tackle the issue of uncertainty by the expert when giving the value for α and finally gain a better result in efficiency value. Table 4 presents the comparison in efficiency value of the proposed α-cut technique versus conventional crisp α-cut. This paper is structured as follows: Section 2 we present the proposed methodology. Result and discussion are presented in Section 3. Section 4 is the conclusion on this paper.

$$\begin{aligned}
Max & \sum_{r=1}^{s} \bar{y}_{rp} \\
s.t. & \sum_{j=1}^{m} \bar{x}_{jp} \\
& \sum_{r=1}^{s} \bar{y}_{rp} - \sum_{j=1}^{m} \bar{x}_{jp} \leqslant 0 \\
& \bar{x}_{ji} \geqslant v_j(\alpha x_{ji}^{b} + (1-\alpha x_{ji}^{a}) \\
& \bar{x}_{ji} \leqslant v_j(\alpha x_{ji}^{c} + (1-\alpha)x_{ji}^{d}), \\
& i = 1,...,n, j = 1,...,m \\
& \bar{y}_{ri} \geqslant u_r(\alpha y_{ri}^{b} + (1-\alpha y_{ri}^{a}) \\
& \bar{y}_{ri} \leqslant u_r(\alpha y_{ri}^{c} + (1-\alpha)y_{ri}^{d}), \\
& i = 1,...,n, r = 1,...s \\
& u_r, v_j \geqslant 0
\end{aligned} \tag{1}$$

II. Proposed Method

The proposed method in finding the efficiency score using Z-number α-cut techniques with trapezoidal membership function consists of six steps.

Step 1. Suppose there are j DMUs with m inputs and s outputs. Let $\widetilde{Z\alpha} = (\widetilde{A\alpha}, \widetilde{B\alpha})$ represents the Z-number α-cut value given by the expert. $\widetilde{A\alpha} = (m, n, o, p)$ is a trapezoidal fuzzy number for the α-cut value with $\widetilde{B\alpha} = (a, b, c, d)$ is a trapezoidal fuzzy number which contain the restriction of certainty (reliability) on $\widetilde{A\alpha}$. Thus, the new parametric linear programming using Z-number α-cut technique can be written as:

$$
\begin{aligned}
&Max \sum_{r=1}^{s} \bar{y}_{rp} \\
&s.t. \sum_{j=1}^{m} \bar{x}_{jp} \\
&\sum_{r=1}^{s} \bar{y}_{rp} - \sum_{j=1}^{m} \bar{x}_{jp} \leqslant 0 \\
&\bar{x}_{ji} \geqslant v_j(\widetilde{Z\alpha} x_{ji}^{b} + (1 - \widetilde{Z\alpha} x_{ji}^{a}) \\
&\bar{x}_{ji} \leqslant v_j(\widetilde{Z\alpha} x_{ji}^{c} + (1 - \widetilde{Z\alpha}) x_{ji}^{d}), \\
&\quad i = 1, ..., n, j = 1, ..., m \\
&\bar{y}_{ri} \geqslant u_r(\widetilde{Z\alpha} y_{ri}^{b} + (1 - \widetilde{Z\alpha} y_{ri}^{a}) \\
&\bar{y}_{ri} \leqslant u_r(\widetilde{Z\alpha} y_{ri}^{c} + (1 - \widetilde{Z\alpha}) y_{ri}^{d}), \\
&\quad i = 1, ..., n, r = 1, ...s \\
&u_r, v_j \geqslant 0
\end{aligned} \tag{2}
$$

Equation 2 is the structure of the new Z-number α-cut technique and the model is not linear.

Step 2. To linearize the model, the second part of the Z-number α-cut technique is added to its first number. In order to do this, the second part of the Z-number is converted to a crisp number by the center of gravity method (defuzzification method) using the following equation:

$$\beta = \frac{\int x \mu_{\tilde{B}}(x) dx}{\int \mu_{\tilde{B}}(x)} (dx) \tag{3}$$

If $\widetilde{\beta} \sim TrFN(a, b, c, d)$, then the center of gravity method is $\frac{a+b+c+d}{4}$. (Zimmermann, 1991)

Step 3. The Z-number model is converted to weighted fuzzy numbers by multiplying β with each element in the trapezoidal fuzzy numbers in the first part which is $\widetilde{A\alpha}$. Figure 1 shows the Z-number and the weighted fuzzy numbers.

Step 4. The weighted fuzzy numbers are then transformed to regular fuzzy numbers while preserving the properties of reliabilities (Azadeh and Kokabi, 2016) by assuming that the slope of its line is equal to the weighted fuzzy numbers. The impact of the slope of the lines on the weighted fuzzy numbers can be seen in Figure 2. Suppose the weighted Z-number, $\widetilde{Z}^{\beta} \sim TrFN(m', n', o', p')$, and its relevant normal fuzzy number has a trapezoidal fuzzy numbers with $\widetilde{N} \sim TrFN(m'', n'', o'', p'')$. Further assume that $n' = n''$ and $o' = o''$ and the slope of the lines are equal. Figure 2 shows the converted weighted fuzzy number.

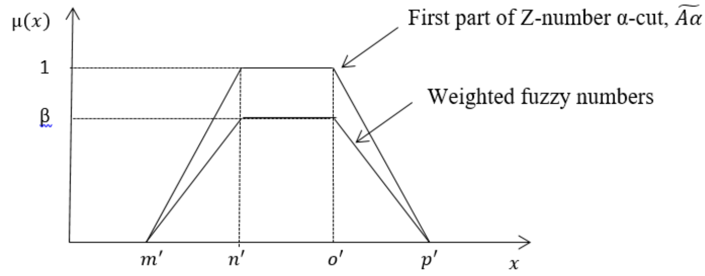

Figure 1: The illustration first part of Z-number an the weighted fuzzy number

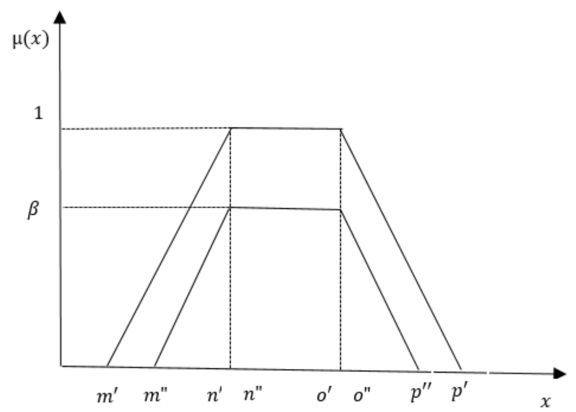

Figure 2: Weighted fuzzy numbers

For $x \leqslant n'$, the linear equation for the normal fuzzy number is given by $\mu_{\widetilde{N}}(x) = \frac{m'}{(n'-m')} x + h$. For finding the value of h, the point of $(n'$,1) is inserted in this equation. Thus :

$$
\begin{aligned}
1 &= \frac{\beta}{(n'-m')}n' + h \rightarrow h = 1 - \frac{(\beta n')}{(n'-m')} \\
\mu_{\widetilde{N}}(x) &= \frac{\beta}{(n'-m')}x + 1 - \frac{\beta n'}{(n'-m')}, x \leqslant n'
\end{aligned} \tag{4}
$$

If $\mu_{\widetilde{N}}(x) = 0$ in equation 4, then the value of m'' is identified by equation 5.

$$
\begin{aligned}
0 &= \frac{\beta}{(n'-m')}m'' + 1 - \frac{\beta n'}{(n'-m')} \\
&\rightarrow \frac{\beta}{(n'-m')}m'' = \frac{\beta n' - n' + m'}{(n'-m')} \\
m'' &= \frac{\beta n' - n' + m'}{\beta}
\end{aligned} \tag{5}
$$

For $x \geqslant o'$, the linear equation for the normal fuzzy number is given by $\mu_{\widetilde{N}}(x) = \frac{\beta}{(o'-p')}x + h$. For finding the value of h, the point of (c,1) is inserted in this equation. Thus :

$$
\begin{aligned}
1 &= \frac{\beta}{(o'-p')}p' + h \rightarrow h = 1 - \frac{\beta o'}{(o'-p')} \\
\mu_{\widetilde{N}}(x) &= \frac{\beta}{(o'-p')}x + 1 - \frac{\beta o'}{(o'-p')}, x \geqslant o'
\end{aligned} \tag{6}
$$

If $\mu_{\widetilde{N}}(x) = 0$ in equation 6, then the value of p'' is identified by equation 7.

$$
\begin{aligned}
0 =& \frac{\beta}{(o'-p')}p'' + 1 - \frac{\beta o'}{(o'-p')} \\
\rightarrow& \frac{\beta}{(o'-p')}p'' = \frac{\beta o' - o' + p'}{(o'-p')} \\
p'' =& \frac{\beta o' - o' + p'}{\beta}
\end{aligned} \tag{7}
$$

Step 5. Normal fuzzy number is then converted to a crisps value by the center of gravity method (defuzzification method) using the following equation:

$$
\Omega = \frac{\int x\mu_{\tilde{B}}(x)dx}{\int \mu_{\tilde{B}}(x)}(dx) \tag{8}
$$

If $\widetilde{\beta} \sim TrFN(m'', n'', o'', p'')$, then the center of gravity method is $\frac{m''+n''+o''+p''}{4}$. (Zimmermann, 1991)

Step 6 : The new α-cut method using Z-number α-cut technique is obtained.

$$
\begin{aligned}
&Max \sum_{r=1}^{s} \bar{y}_{rp} \\
&s.t. \sum_{j=1}^{m} \bar{x}_{jp} \\
&\sum_{r=1}^{s} \bar{y}_{rp} - \sum_{j=1}^{m} \bar{x}_{jp} \leqslant 0 \\
&\quad \bar{x}_{ji} \geqslant v_j(\Omega x_{ji}^{b} + (1-\Omega x_{ji}^{a}) \\
&\quad \bar{x}_{ji} \leqslant v_j(\Omega x_{ji}^{c} + (1-\Omega) x_{ji}^{d}), \\
&\quad\quad i = 1, ..., n, j = 1, ..., m \\
&\quad \bar{y}_{ri} \geqslant u_r(\Omega y_{ri}^{b} + (1-\Omega y_{ri}^{a}) \\
&\quad \bar{y}_{ri} \leqslant u_r(\Omega y_{ri}^{c} + (1-\Omega) y_{ri}^{d}), \\
&\quad\quad i = 1, ..., n, r = 1, ...s \\
&\quad\quad u_r, v_j \geqslant 0
\end{aligned} \tag{9}
$$

III. Results and Discussion

The data is taken from (Ibrahim, 2019) on portfolio selection in IS/IT (Information Systems/Information Technology) in one of the national govermental organization (Table 1). This study focuses on finding the efficiency of 16 projects. The efficiency score is measured by Z-number CCR using trapezoidal fuzzy numbers method (Ibrahim, 2019) provided the α-cut is in the form of Z-number instead of crisps number. The score that is near or equal to 1 is considered as a good portfolio. Each project is labelled as a DMU with the input criteria is the cost of the project ($ million) while the outputs are number of potential subsequent investments, contribution to the workflow improvement and the percentage of contribution to electronic readiness. The inputs and outputs values are evaluated by the experts (see Table 1) with the membership function parameters for the reliability judgement on the data is in Table 2. Using the method outlined in Section 2 with α-cut value given by expert is in Z-number form, $\widetilde{Z\alpha} = (\widetilde{A\alpha}, \widetilde{B\alpha})$, trapezoidal fuzzy numbers for $\widetilde{A\alpha} = (0.48, 0.49, 0.50, 0.51)$ is assumed with a medium confidence level (reliability) to the α-cut value given, $\widetilde{B\alpha} = (0.65, 0.75, 0.85, 0.95)$, the efficiency values for each project together with the result of conventional FDEA model is presented as in Table 3. The results are different as the proposed method include the uncertainty information of the expert towards his/her judgement. The features of the proposed method and the existing methods is compared to show the propose method improvement (Table 4).

IV. Conclusion

There are four traditional approaches in solving fuzzy DEA model which are the fuzzy ranking approaches, the tolerance approach, the defuzzification approach and the α-cut based approach. The most popular approach used by researchers is the α-cut based approach as it has a simpler calculation. However, the drawback of the traditional α-cut based approach is that the α-cut value is a crisps value given by the experts and it does not include the uncertainty of the expert judgement. In this paper, we proposed the Z-number α-cut technique using the trapezoidal fuzzy numbers that includes more uncertainty information on the judgement given by the experts. A numerical example on portfolio selection in IS/IT (Information Systems/Information Technology) is presented to demonstrate the proposed method and to the efficiency score of the portfolios. The efficiencies score obtained are more reliable compared to the existing α-cut method as it contains more informations by the experts.

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Table 1: Trapezoidal fuzzy numbers assigns to each inputs and outputs.

Project number	Cost of the project ($ million) (Input 1)	No. of potential subsequent investments (Output 1)	Contribution to the workflow improvement (Output 2)	Percentage of contribution to electronic readiness (Output 3)
1	(412, 427.3, 442.6, 458)	(128, 130.7,133.7, 136)	(0.73, 0.82, 0.91, 1)	(42, 44.7, 47.4, 50)
2	(174, 176.7, 179.4, 182)	(69, 73, 77, 81)	(0.05, 0.13, 0.21, 0.29)	(6, 8, 10, 12)
3	(225, 236.3, 247.6, 259)	(27, 27.7, 28.4, 29)	(0.68, 0.76, 0.84, 0.91)	(36, 39.3, 42.6, 46)
4	(308, 318, 328, 338)	(85, 85.3, 88.6, 95)	(0.55, 0.65, 0.75, 0.85)	(87, 89, 91, 93)
5	(175, 184.3, 193.6, 203)	(73, 74.3, 75.6, 77)	(0.37, 0.47, 0.57, 0.68)	(71, 73.7, 76.4, 79)
6	(84, 90, 96, 102)	(66, 68.7, 71.4, 74)	(0.07, 0.15,0.23, 0.31)	(45, 46.3, 47.6, 49)
7	(349, 363, 377, 391)	(123, 127.7, 132.4 137)	(0.95, 0.99, 0.99, 0.99)	(39, 42.3, 45.6, 49)
8	(245, 249, 273, 297)	(41, 42.3, 43.6 , 45)	(0.31, 0.40, 0.49, 0.59)	(32, 35.3, 38.6, 42)
9	(151, 153, 155, 157)	(58, 59.3, 60.6, 62)	(0.35, 0.41, 0.47, 0.65)	(25, 26.3, 27.6, 29)
10	(265, 275.7, 286.4, 297)	(49, 51, 53, 55)	(0.68, 0.76, 0.86, 0.94)	(37, 39.7, 42.4, 45)
11	(345, 356.3, 367.6, 379)	(21, 23, 25, 27)	(0.15, 0.17, 0.19, 0.21)	(54, 56.7, 59.4, 62)
12	(215, 219.7, 224.4 , 229)	(4, 4.7, 5.4, 8)	(0.19, 0.196, 0.203, 0.21)	(56, 58, 60 , 62)
13	(385, 389, 394, 397)	(6, 7.3, 8.6, 10)	(0.33, 0.337, 0.343, 0.35)	(34, 35.3, 36.6, 38)
14	(454, 467.3, 480.6, 494)	(7, 8.3, 9.6, 11)	(0.44, 0.46, 0.48, 0.50)	(11, 12.3, 13.6, 15)
15	(384, 388, 392, 396)	(7, 7.7, 8.4, 9)	(0.20, 0.21, 0.23, 0.24)	(48, 50, 52,54)
16	(384, 388.7, 393.4, 398)	(9, 10.3, 11.6, 13)	(0.16, 0.18, 0.20, 0.22)	(52, 53.3, 54.6, 56)

Table 2: Reliability values given by experts to the projects criteria.

Z=(A, B)		Membership functions parameters
B	High	(0.8, 1, 1, 1)
	Medium	(0.65, 0.75, 0.85, 0.95)
	Low	(0.5, 0.6, 0.7, 0.8)

Table 3: Results of the proposed model and the FDEA model

Project number	FDEA Eff. score	Proposed model Eff. score
1	0.6034643	0.8990417
2	0.5593223	0.9323193
3	0.5873105	1
4	0.5309568	0.9963907
5	0.6623612	1
6	1.0000000	1
7	0.5528586	0.9608039
8	0.3685344	1
9	0.9887899	1
10	0.5078825	0.7983467
11	0.3506480	0.7061635
12	0.5079757	1
13	0.2122091	0.3895661
14	0.1304303	0.2762829
15	0.1627877	0.3816038
16	0.1805543	0.4883014

Table 4: Features of proposed model versus other studies.

Model	Category of Numbers	Fuzzy Approximation	α-cut Approach
CCR Model (4)	Crisps	-	-
FDEA Model (11)	Fuzzy	Triangular Fuzzy Numbers	Crisps
Z-Number CCR Model using Triangular Fuzzy Numbers (2)	Z-Number	Triangular Fuzzy Numbers	Crisps
Z-Number CCR Model using Trapezoidal Fuzzy Numbers (7)	Z-Number	Trapezoidal Fuzzy Numbers	Crisps
The Proposed Model	Z-Number	Trapezoidal Fuzzy Numbers	Z-Number