ASM Sc. J., 12, Special Issue 1, 2019 for IQRAC2018, 276-284

Dynamical Process On Growing Geometrical Network Based On Modular Group

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Many network models have been proposed and constructed to mimic the underlying features of complex networks. Studying the dynamical process of a network gives a good platform to understand how the underlying geometrical and structural features influence various transport properties. In this study, the dynamical process on the network is described by using random walks. From this process, some of the random walk transport properties are determined such as relaxation time, mean first passage time (MFPT), random walk centrality (RWC), average trapping time (ATT) and global mean first passage time (GMFPT). We find that GMFPT grows exponentially when the network grows. This is mainly due to some central nodes that have high RWC, which tends to attract the random walker more compared to a node with a lower RWC. This study plays an important role in determining the performance of the network.
Keywords: Complex networks, Mean first passage time, Random walk, Random walk centrality

I. Introduction

The world is abundant of complex networks in various field from biology, physics to computer science. One of the earliest network models is introduced by Watts and Strogatz (1998) which shows that most of the real-world networks exhibit features like small-world network that has high clustering coefficient. Another type of network model is scale-free (SF) network based on Albert et al. (1999) that follows power-law degree distribution $P_{deg}(K) \sim K^{(-\gamma)}$ where $P_{deg}(K)$ is the fraction of nodes with the number of links attached to it known as degree, K. To have a scale-free property, usually the power law exponent, γ is at the range of $2 \leqslant \gamma \leqslant 3$. SF network is an example of heterogeneous network and it has significant effect on physical problems where it is stable in opposition to the removal of nodes but fragile under attacks aiming on nodes with high degree (Cohen et al., 2000). Some of the real-life examples of SF network are the World Wide Web (WWW) and the Internet. Other types of network models that are prepared to mimic the real-world system are the hyperbolic network model (Aste et al., 2005) and the growing geometrical network (Wu et al., 2015). Although the study on these models is mainly focusing on the static properties of the networks, the interest in the dynamical part is growing. The dynamical process can be described using random walk on the network since they have been used in the dynamic of many natural and artificial systems such as fluctuations in stock market (Fama, 1995) and natural disasters (Wijesundera et al., 2016). Random walk (RW) is basically a Markov process where each step is independent of the previous event. Some of the properties of RW that have been studied broadly are relaxation time (τ), mean first passage time (MFPT) from one node to another node and the random walk centrality (RWC).

MFPT, $\langle T_{ij} \rangle$ is defined as the time needed for a walker to reach a node from a starting point and this property is widely used in characterizing transport efficiency (Kozak and Balakrishnan, 2002, Zhang et al., 2009, 2010) It was shown that MFPT behaves differently depending on the topology. For example, it behaves sublinearly with the size of network, N in some scale-free networks while behaves very differently in standard regular fractals (Bentz et al., 2010, Lin et al., 2010) where it scales superlinearly with N. MFPT plays an important role in determining the RWC which describes centralization of information travelling over networks and also the average trapping time $\langle T_j \rangle$, which is the average of MFPT to the trap node j taken over all starting point. Besides, with the calculation of MFPT, global mean first passage time (GMFPT) for the whole network can also be determined.

In this paper, we compute MFPT on a growing geometrical network (GGN) constructed from tessellation of modular group (Taha et al., 2016). Based on the calculated MFPT value, we can explain how the walker behaves in GGN via the study of transport properties such as RWC, relaxation time, ATT as well as GMFPT. Besides, by studying these transport properties, we can shed some light about the structure of the network either it being a homogeneous or a heterogeneous network.

A. Random Walks on Network

To apply the RW, first, we consider the GGN network, represented as $G(V, E)$ where V is the set of nodes while E is the set of undirected edges. The connection of two nodes i and j can be represented by adjacency matrix, $\mathbf{A}$ which equals to 1 if there is a relation between them and 0 if otherwise. Since the GGN is an undirected network, it has the property of $A_{ij} = A_{ji}$. For degree of node i, it is given as $K_i = \sum^j A_{ij}$. As for the walker, since it cannot remain at the same node, it will move to another node once at a time with the probability of $\frac{1}{K_i}$. Using the adjacency matrix and the probability, we can determine the transition probability which is defined by the walker movements. If the walker is on node i at time t chooses one of its neighbour with equal probability at time $t+1$, the transition probability can be written as $W_{ij} = \frac{A_{ij}}{K_i}$. Since the walker starts at node i at time $t = 0$ and arrived at node j at time t; the master equation can be expressed as

$$P_{ij}(t+1) = \sum_k \frac{A_{kj}}{K_k} P_{ik}(t). \tag{1}$$

Since there is no boundary in network, the random walker can move in any direction inside the GGN. However, it cannot leave the system and this is similar to the diffusion process except that all walkers must follow the equation at every time step (Lau and Szeto, 2010). In order to find the walker probability at j from i at t, the asymptotic behaviour of the transition probability is considered. Based on the principle of detailed balance with $t \to \infty$, the system reaches a state where there is no net flow of random walker in any direction. This leads to $K_i P_j^\infty = K_j P_i^\infty$ where

$$P_j^\infty = \lim_{t\to\infty} P_{ij}(t).$$

The equilibrium state can be defined as

$$P_j^\infty = \frac{K_j}{2L}. \tag{2}$$

where L denotes the number of edges. Based on Eq.2 at equilibrium state, the walker has a higher tendency to move to nodes with more edges. In the other hand if the system's net flow is not zero, it will consist of both equilibrium state probability and fluctuation around it.

In order to figure out the MFPT, we need to find a relation between first passage time, $F_{ij}(t)$ and probability of the walker to reach the destination, $P_{ij}(t)$. This relation described the time step of a random walker where it continues to move even when the destination node is reached at time t' due to the time step is set at t, then returns to j after $t - t'$ steps (Redner, 2001). Thus, the first-passage probability satisfies the relation

$$P_{ij}(t) = \delta_{ij}\delta_{t0} + \sum_{t'-0}^{t} F_{ij}(t)P_{jj}(t - t'). \quad (3)$$

where $\delta_{ij}\delta_{t0}$ represents to the initial condition of the walker probability. This equation can be decoupled by using Laplace transform (Hughes, 1995, Redner, 2001) and yields

$$\tilde{P}_{ij}(s) = \delta_{ij} + \tilde{(F)}_{ij}(s)\tilde{P}_{jj}(s). \quad (4)$$

The details of the formulation of MFPT are explained in Samsul et al. (2018). By expressing the terms in Eq.4 with Laplace transform, MFPT, $\langle T_{ij}\rangle$ is obtained as follows

$$\langle T_{ij}\rangle = \begin{cases} \frac{2L}{K_j}\left[R_{jj}^{(0)} - R_{ij}^{(0)}\right] & \text{for} j \neq i \\ \frac{2L}{K_j} & \text{for} j = i \end{cases} \quad (5)$$

Other than MFPT, another transport property of random walk that can be identified here is the relaxation time, τ where it is the asymptotic time of convergence to the equilibrium or stationary distribution. It is defined as

$$\tau_j = R_{jj}^{(n)} \equiv \sum_{t=0}^{\infty}\left[P_{jj}(t) - P_j^{\infty}\right]. \quad (6)$$

Another measurement called random walk centrality can also be obtained when the random walk motions are asymmetric, $i \neq j$. The measurement equation is given by

$$C_j \equiv \frac{P_j^{\infty}}{\tau_j}. \quad (7)$$

It measures the speed of the walker to move from one node to another node. The walker will reach the target node earlier in consequence of high RWC value. This property plays different effects depending on the type of network. For example, in terms of communication, a node with high RWC will receive signal emitted by its partner earlier (Noh and Rieger, 2004). Using the MFPT computed from Eq.5, we can study the trapping problem defined on GGN. Let $\langle T_j\rangle$ be the average trapping time, or the average of MFPT, $\langle T_{ij}\rangle$ to the trap node j, taken over all starting point. The equation $\langle T_j\rangle$ is expressed as

$$\langle T_j\rangle = \frac{1}{N-1}\left(\sum_{j=1}^{N}\langle T_{ij}\rangle - \langle T_{ii}\rangle\right). \quad (8)$$

For the GMFPT, $\langle T_g\rangle$, it is defined as average over all the trap nodes for the network. The formulation of $\langle T_g\rangle$ is expressed by

$$\langle T_g\rangle = \frac{\langle T_j\rangle}{N}. \quad (9)$$

II. Methodology

The growing geometrical network in this study is constructed based on using modular group, a discrete subgroup of $PSL(2, R)$ tessellating the hyperbolic plane. The tessellation is generated with Mathematica by using linear fractional transformation and can be found in Taha et al. (2016). As number of iterations increases, the size of the network grows exponentially producing large number of nodes and edges. This is due to the growth of concatenation of words or generators of the modular group (Taha et al., 2016). Table 1 shows the total number of nodes and edges produced from different iteration while Figure 1 shows the constructed network.

To compute the value of MFPT based on Eq.5, a matrix formulation needs to be formed first by using equation $R_{ij}^{n} \equiv \sum_{t=0}^{\infty}[P_{ij}(t) - P_j^{\infty}]$. The equation can be written as

$$\mathbf{R}^{(0)} = \sum_{t=0}^{\infty}(\mathbf{W}^t - \mathbf{Q}). \quad (10)$$

where $\mathbf{R}^{(0)}$ is similar to $R_{ij}^{(0)}$. Matrix $\mathbf{W}$ in Eq.10 represents the transition matrix and $\mathbf{Q} \equiv \mathbf{P}^{\infty}\mathbf{1}$ represents the equilibrium probability matrix. A relation of $\mathbf{Q}^n = \mathbf{Q}$ for $n > 0$ and $\mathbf{WQ} = \mathbf{QW} = \mathbf{Q}$ are the consequence of the matrix Q being the projection of W onto the

Table 1: Total number of nodes and edges produced from different iteration.

Number of iteration, i	Number of nodes	Number of edges
1	6	8
5	23	44
10	100	207
15	414	872
20	1695	3585

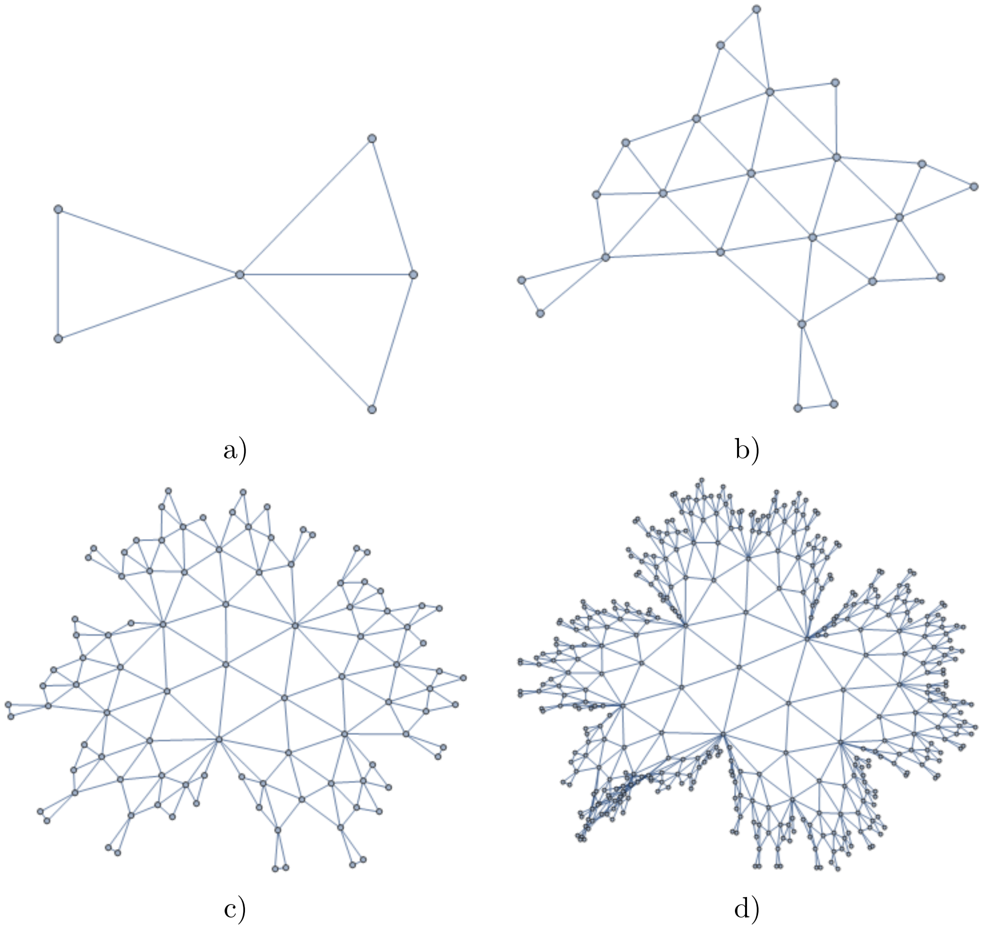

Figure 1: Underlying network from the modular group tessellation with iteration a)$i = 1$, b)$i = 5$, c)$i = 10$ and d)$i = 15$

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subspace with eigenvalue 1 (Noh, 2007). From these relations, $\mathbf{W}^n - \mathbf{Q} = (\mathbf{W} - \mathbf{Q})^n$ for $n > 0$ and $(\mathbf{I} - \mathbf{Q})$ for $n = 0$. Applying all of them into Eq.10, it can be redefined as

$$\mathbf{R}^0 = \frac{1}{\mathbf{I} + \mathbf{Q} - \mathbf{W}} - \mathbf{Q}. \qquad (11)$$

To determine both MFPT and RWC, Eq.11 needs to be evaluated and the result is substituted into the Eq.5 to yield the MFPT. This step is repeated for every iteration for all constructed network from the modular group.

III. Results and Discussion

Computation of MFPT has been carried out on networks constructed from tessellation of modular group on a hyperbolic plane with different sizes. Computing MFPT enables us to know what is the mean time for a random walker to reach a particular node for the first time. With the MFPT values for every node in a particular network, we can compute the average trapping time (ATT), $\langle T_j \rangle$ for a particular trap node j taken over all starting points in the network. ATT is important as it can be used to characterize the network structural properties based on transport efficiency. One example of trapping problem is the work of Montroll (1969) in the application to excition trapping on photosynthesis units.

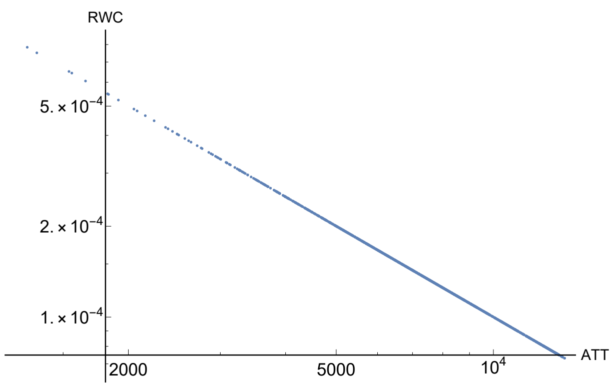

Figure 2: Log-log plot of random walk centrality vs average trapping time for modular group at 20^{th} iteration.

Figure 2 shows the graph plot of RWC against the ATT for every node in 20^{th} iteration GGN. Table 1 shows the number of nodes increases as the iteration, i increases. $i = 1, 5, 10, 15, 20$ are chosen generally to reflect on how the network grows. However, we generate the network model for each iteration between $1 \leq i \leq 20$. According to Table 1, a 20^{th} iteration GGN has 1695 nodes and 3585 number of edges. RWC is decreasing linearly with the ATT which implies that the longer the time needed for the walker to arrive at certain nodes, the less important that the node becomes in the network. For example, a less important node in the network will be less visited by the walker in comparison to the other nodes. Say we have two nodes where $C_j > C_i$, the random walker that starts at node i will reach j earlier compared to when it started on node j and wants to reach i. Thus a node with a larger RWC tends to be visited earlier by the random walker rather than a node with smaller RWC. In another word, a node with a high valuer of RWC will attract the walker to visit it more frequently. Also, a high value of RWC in a node indicates that the node is important in diffusion or trapping process.

Structure of the network plays an important role in determining the values of RWC and ATT for a particular node. In Tables 2 and 3, we identify the highest and lowest ten values of ATT and RWC for nodes in the GGN. The location of these nodes is also highlighted in the network as shown in Figure 3. Nodes with the highest values of ATT (red in colour) are all located at the periphery part of the network. On the other hand, nodes with lowest values of ATT (green in colour) are all located at the center of the network. This shows that the central part of the network or the "freeze" region has the most effective communication between nodes while the outer regions are pretty much less effective.

If we look at the degree of the nodes on the network, they are distributed between the range of $2 \leq K \leq 18$. Nodes with the highest values of ATT are all having value of $K = 2$. Meanwhile, the nodes with lowest values of ATT possess the value of K ranging from $6 \leq K \leq 18$. From the static process, by looking at the degree distribution, the network can be identified as a heterogeneous network because the degree is not distributed evenly. From the dynamical process, RWC or ATT values can be used to explain this matter as their distribution shows asymmetry in the network.

From the same GGN, we also study the relaxation time, τ. Figure 4 shows how the relaxation time behaves with the node degree of the network. We examined the distribution of relaxation time by calculating the τ for each

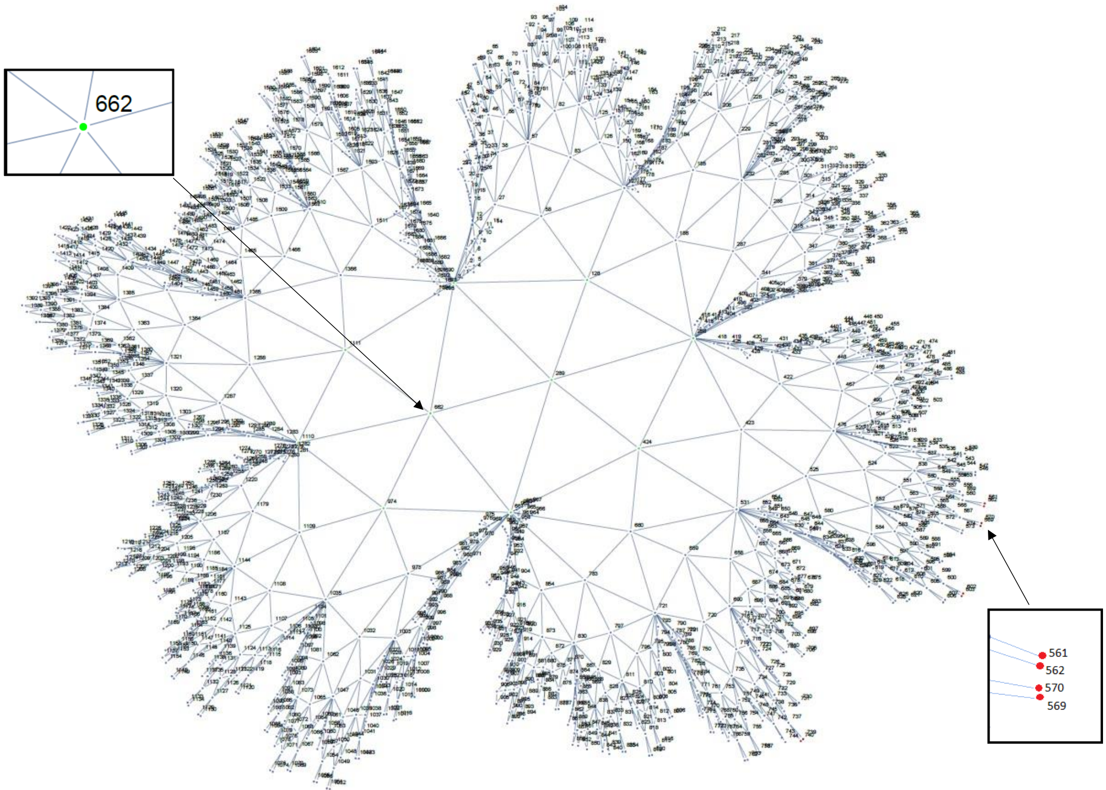

Figure 3: GGN at 20^{th} iterations where the magnified points show the highest and the lowest values of ATT.

Table 2: Lowest values of ATT.

Nodes	ATT	RWC	K
661	1280.41	0.000783507	18
3	1335.20	0.000750921	18
288	1539.2	0.000651432	17
289	1558.59	0.000643515	6
662	1655.63	0.000605669	6
424	1824.68	0.000549441	6
1119	1832.4	0.000546916	17
128	1913.76	0.000523787	6
975	2049.37	0.00048903	6
974	2078.804	0.000482091	6

Table 3: Highest values of ATT.

Nodes	ATT	RWC	K
739,740	13687.5	0.00007308	2
602,603	13569.9	0.000073714	2
569,570	13534.5	0.0000739081	2
561,562	13475.7	0.0000742325	2
332,333	13363.4	0.0000748545	2

K. A short relaxation time in random walk is known as random walks with non-compact exploration (Bénichou et al., 2010, Hwang et al., 2012) whereas a high value of τ indicates a longer time is spent by the random walker traveling around at the neighbourhood nodes (Lee et al., 2014). This phenomenon is called a compact case and a non-compact case behaves on the contrary. Based on Figure 4, there is an obvious variation of the value of τ ranging from the lowest τ to the highest at the node with

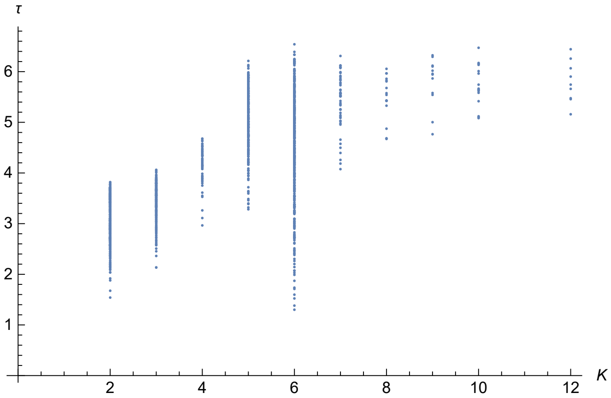

Figure 4: τ vs K for the 20^{th} iteration GGN.

degree of six. This is because node with degree six is usually nodes that form the central part (saturated part) of the network and they appeared the most in the network. These nodes are not only appeared at the central region, they also appeared in different region of the network. Hence, the relaxation time turns out to be distributed broadly between $1 \leq \tau \leq 7$. On the contrary, another region with different value of K has much smaller range of τ due to active boundary of the network (Wu et al., 2015) where it can still link many triangles.

According to Eq.7, τ is inversely proportional to RWC. When RWC is high, the relaxation for that particular node is low. This implies that most nodes in the central regions have low relaxation time which means that they can converge to stationary distribution easily. If we consider the walker as information, it would mean that the information can be supplied to all nodes in that region in a short time.

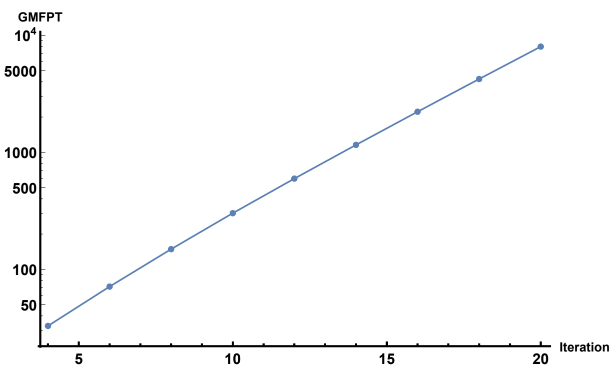

Figure 5: GMFPT vs Iteration.

Then, we compute the GMFPT of every other iteration from the modular group. Figure 5 shows a semi-log plot between GMFPT and iterations and it shows that GMFPT grows exponentially. This trend indicates that it has a power-law function. The linear scaling of GMFPT with the iteration also shows that the underlying structure of the network is heterogeneous and not homogeneous. This trend can also be observed in complete graphs (Bollt and ben Avraham, 2005). For having heterogeneous structures, this means that there are central nodes that gas very large degree. These nodes usually have high RWC and the walkers are attracted to them.

High value of GMFPT indicates a longer time for the walker to cover the whole network, thus affects the efficiency of the network model. However, when compared to a homogeneous network (random network), the GMFPT is much lower.

IV. Conclusion

In summary, we have discussed five transport properties of random walk process in growing geometrical network. From the computed MFPT, we managed to determine the relaxation time, τ, random walk centrality, average trapping time and global mean first passage time. RWC and ATT have revealed that GGN of interest has heterogeneous structure due to asymmetry in dynamics. As for the relaxation time, we found that the central part (saturated part) of the network has the lowest value. This implies that the central regions have a fast converging time. From the global perspective, the linear scaling of GMFPT indicates the structure of the networks has a scale-free property.

Acknowledgments

The authors wish to acknowledge the financial support provided through the Fundamental Research Grant Scheme (FRGS), Project No. 01-01-16-1865FR by Ministry of Higher Education.

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