ASM Sc. J., 12, Special Issue 1, 2019 for IQRAC2018, 310-319

The TSR-MM Based on Robust Location and Scales Measures in Dual Response Optimization in the Presence of Outliers and Heteroscedastic Errors

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The dual response surface optimization approach is commonly used in an industrial process to simultaneously optimize the process sample mean and the process sample standard deviation functions. The shortcoming of this approach is that the sample mean and the sample variance are used to fit the process mean and process variance functions based on the OLS method. However, these estimators are very sensitive to outliers or departures from the normality assumption. The OLS estimates do not give good results when both outliers and heteroscedastic errors exist concurrently. As a consequence, the optimum operating conditions may be located far from the true optimum values. In order to make significant improvements in robust design studies, robust location (median) and robust scales estimates (Median Absolute Deviation (MAD) and Interquartile Range (IQR)) of the response variables are employed for dual response surface optimization. Two-stage robust estimator based on MM-estimator (TSR-MM based) based on robust location and robust scales estimates is proposed to simultaneously remedy the problem of heteroscedastic errors and outliers. The results of the study indicate that the TSR-MM based on robust location and scales estimates provide a significant reduction in the bias and variance of the estimated mean response.

Keywords: dual response surface model, outliers, optimization, robust location, robust scales, robust MM-estimator

I. Introduction

During the last few decades, much of response surface methodology (RSM) was focused on finding the operating conditions that resulted in an optimum response of the mean with the homogeneity assumptions on the variances. With the challenge of economic, industrial statisticians have become aware that they can no longer focus only on the optimal process mean of the response of interest. Instead, the process variance of the response also needs to be considered. As noted by (Vining and Myers, 1990), the common problem in an industrial process is to minimize the process variability that is inherently part of some process, which can achieve the target value for characteristics of interest. The dual response surface optimization approach, introduced by (Vining and Myers, 1990), is a useful technique to monitor an industrial process by using both the mean and the standard deviation of the measurements as the responses. In practice, the two separate models give a more understanding analyst of the optimization process and thus allow them to see what levels of the control variables can lead satisfactory values of the response as well as the variance.

Like other optimization work in RSM, the method of least squares is widely used to solve the process target problem. With the assumption of OLS to retain the error variance to be homogeneous, it is quite difficult in a certain situations and thus resulting in heteroscedasticity. In such a situation, the iterative weighted least squares (IWLS) estimation procedure is often used to estimate the responses when heteroscedasticity occurs in the data (Montgomery et al., 2001). The IWLS is an alternative method to finding a transformation that stabilizes the response, Y. However, the IWLS estimators suffers a huge set back in the presence of a few a typical observations that we often call outliers (Midi et al., 2009).

The model based on robust techniques for improving the quality of the process is successfully applied to different industrial problems. Nonetheless, in the presence of both outliers and heteroscedasticity in response surface model, not much work has been reported in the literatures for estimating the parameters. (Goethals and Cho, 2011) proposed the Reweighted Least Squares (hereafter we denote this procedure as RLS) to handle the problems of heterocedasticity in response surface model. Their work did not take into consideration when outliers are present in the data. This problem has motivated us to develop a new and more efficient estimator that can rectify these two problems simultaneously. In this paper, we propose an alternative method to handle heteroscedastic problem in the existence of outlier. The proposed method incorporates the highly efficient and high breakdown point estimator, specifically the MM-estimator in the formulation of robust design in response surface model. Real data sets and Monte Carlo simulations support our view that the newly proposed robust weighted least squares method outperforms the existing estimation techniques in the presence of heteroscedasticity and outliers.

II. Methodology

A. Estimation of Heteroscedastic Regression Model

Consider a heteroscedastic regression model:

$$\begin{aligned} y_i &= {x_i}^T\beta + \epsilon_i \\ i &= 1,2,K,n \\ E(\epsilon_i|x_i) &= 0 \\ E({\epsilon_i}^2 x_i) &= \sigma_i \end{aligned} \qquad (1)$$

where the y_i is the response variables, x_i are known $n \times 1$ design vectors of independent variables, β is an unknown vector of interest, and ϵ_i is the component of an $n \times 1$ random vector. Assuming $Y = (y_1, y_2, K, y_n)^T$, $x = (x_1, x_2, K, x_n)^T$, $\epsilon = (\epsilon_1, \epsilon_2, K, \epsilon_n)^T$, and $\Omega = diag({\sigma_1}^2, K, {\sigma_n}^2)$, then equation (1) can be written as

$$\begin{aligned} Y &= X\beta + \epsilon \\ E(\epsilon X) &= 0 \\ E(\epsilon^T \epsilon X) &= \Omega \end{aligned} \qquad (2)$$

Here, without loss of generality, we assume that X is a column full rank matrix, i.e. $rank(X) = k$. The OLS estimators of regression coefficients introduced is used, and the parameter $\hat{\beta}$ is defined as $\hat{\beta} = (X^TX)^{-1}X^Ty$ is the best linear unbiased estimator with

$$var(\hat{\beta}) = (X^TX)^{-1}X^T\Omega X(X^TX)^{-1} \qquad (3)$$

where $E(\hat{\epsilon}\hat{\epsilon}^T) = \Omega$, a positive definite matrix. If the errors are homoscedastics, that is $\Omega = \sigma^2 I$, Equation (2) simplifies to $var(\hat{\beta}) = \hat{\sigma}^2(X^TX)^{-1}$. If the errors are heteroscedastic, that is $\Omega = \sigma^2 V$, the Equation (2) becomes

$$var(\hat{\beta}) = \sigma^2(X^TX)^{-1}X^TVX(X^TX)^{-1} \qquad (4)$$

When heteroscedasticity in the residuals is encountered, transformation of variables may be conducted as remedial procedure (Goethals and Cho, 2011). The weighted least squares (WLS) is often used to remedy the problem of non-constant error variances (Kutner et al., 2004). The weighted least squares estimator and maximum likelihood estimates of the regression for a model with k parameter can be expressed as

$$\hat{\beta}_{WLS} = (X^T W X)^{-1} X^T W y \quad (5)$$

where

$$var(\hat{\epsilon}) = W^{-1} = \begin{bmatrix} \frac{1}{V_1} & 0 & \Lambda & 0 \\ 0 & \frac{1}{V_2} & \Lambda & 0 \\ M & M & O & M \\ 0 & 0 & \Lambda & \frac{1}{V_n} \end{bmatrix} \quad (6)$$

and $w_i = \frac{1}{\hat{v}_i}$, where $\hat{v}_i$ is the fitted values and $var(\hat{\beta}_{WLS}) = \sigma^2{}_{WLS}(X^T W X)^{-1}$ where $\sigma^2{}_{WLS} = MSE_{WLS} = \frac{\sum_{i=1}^{n} w_i \hat{\epsilon}_i{}^2}{n-p}$.

B. The Reweighted Least Squares

In the situation where heteroscedasticity errors exist, the OLS is not an efficient estimator. (Kutner et al., 2004) proposed weighted least squares to estimate the parameters of multiple linear regression models for solving the error variance which is not constant. (Goethals and Cho, 2011) employed this technique and integrated it in the response surface methodology. They refer this technique as Reweighted Least Squares (RLS). The RLS-based method is defined in two stages as follows:

Stage 1.

(i) Fit model in Equation (1) by the ordinary least squares to obtain $\hat{\beta}_{OLS}$.

(ii) Compute the residuals $\hat{\epsilon}_i = y_i - f(x_i; \hat{\beta}_{OLS})$ and then regress the absolute residuals, denoted as s_i where $s_i = |\hat{\epsilon}_i|$, on $\hat{y}_i$ also by using ordinary least squares method.

(iii) Find the fitted values $\hat{s}_i$ from the stage 1 (ii).

(iv) The square of the inverse fitted values would form the initial weight, $w_{ii} = \frac{1}{\hat{s}_i^2}$.

Stage 2. Solve for new weighted least squares estimates

$$b^{(t)} = \left(x' w^{t-1} x\right)^{-1} x' w^{t-1} y$$

where x is the model matrix, diagonal current weight matrix, and t is a number of iteration. Stage 1 and Stage 2 are repeated until the estimated coefficients converge.

(Goethals and Cho, 2011) employed the RLS methods to estimate the parameters of the second-order polynomial models for the process mean ($\bar{y}$) and process standard deviation (s) of the response y. The fitted response functions for the process mean and process standard deviation are as follows:

$$\hat{\mu}(x) = X\hat{\beta}_{\mu} \text{ and } \hat{\sigma}^2(x) = X\hat{\beta}_{\sigma^2}$$

where

$$\hat{\beta}_{\sigma^2} = (X^T W X)^{-1} X^T W \bar{y}$$

and

$$\hat{\beta}_{\mu} = (X^T W X)^{-1} X^T W s^2$$

$\bar{y} = [\bar{y}_1, \bar{y}_2, K, \bar{y}_n]^T$ and $s^2 = [s^2{}_1, s^2{}_2, K, s^2{}_n]^T$

C. Two-Stage Robust Weighted Least Squares Estimator

(Kutner et al., 2004) proposed two-stage procedure to rectify the problem of non-constant variance or heteroscedasticity. (Goethals and Cho, 2011) referred to this procedure as the RLS procedure. The main limitation of this estimator is that it depends on the classical estimation technique to estimate the parameters of a model. It is now evident that the OLS method is easily affected by outliers (Midi et al., 2009,). Although the RLS based method can rectify the heteroscedastic error, but it is not robust when outliers occur in the data. In this situation, the RLS based method cannot handle both problems at the same time. We need to improve this method that can remedy the problem of heteroscedastic errors and dampen the effects of outliers. In this respect, we propose to incorporate robust MM estimator in the formulation of the Two-Stage Robust (TSR-MM based) procedure. The proposed algorithm for the TSR-MM based is similar to the RLS except that all the classical estimation techniques used to estimate the parameters in each stage are replaced by the robust estimators. We also modify at stage 2 of the RLS by adapting the MM procedure to the transformed model. The TSR-MM algorithm developed consists of the following two stages.

Stage 1.

(i) Fit model in Equation (1) by using MM estimator (Yohai, 1987) to obtain $\hat{\beta}_{MM}$.

(ii) Regress the absolute residuals, denoted as s_i where $s_i = |\hat{\epsilon}_i|$, (obtained from Stage 1 (i)) on $\hat{y}_i$ also by using the MM-estimator.

(iii) Find the fitted values $\hat{s}_i$ from the stage 1 (ii).

(iv) The square of the inverse fitted values would form the weight, $w_i = \frac{1}{\hat{s}_i^2}$.

Stage 2.

Following (Riazoshams et al., 2010), the two-stage robust (TSR-MM based) estimator is obtained by adapting the MM estimator to the model (1) after it has been transformed.

$$y_i{}^* = y_i \times w_i,\ f_i{}^*(x_i, \beta) = f_i(x_i, \beta) \times w_i$$

and

$$\epsilon_i{}^* = \epsilon_i \times w_i$$

The $\hat{\beta}_{TSR-MM}$ of the MM is obtained by

$$\min_{\hat{\beta}_{TSR-MM}} \sum_{i=1}^{n} \rho\left(\frac{y_i{}^* - f^*(x_i;\beta)}{\sigma}\right)$$

.

Similar to the approach by (Goethals and Cho, 2011), TSR-MM based approach is utilized to estimate the parameters of the function for the process mean and process standard deviation.

D. Robust Design Optimization Procedure in Dual Response Model Based on Two-Stage Robust Weighted Least Squares

Let $\hat{\mu}(x)$ and $\hat{\sigma}(x)$ represent the fitted response surface function for the sample mean and sample standard deviation defined in Section B. (Step 2). We formulate the new fitted response function for the process mean and process variance based on TSR-MM estimator already developed in Section C..

$$\hat{\mu}(x) = b_{0(TSR-MM)} + x^{'} b_{TSR-MM} + x^{'} B_{TSR-MM} x$$

$$\hat{\sigma}^2(x) = c_{0(TSR-MM)} + x^{'} c_{TSR-MM} + x^{'} C_{TSR-MM} x$$

where $b_{0(TSR-MM)}$, b_{TSR-MM}, B_{TSR-MM}, $c_{0(TSR-MM)}$, c_{TSR-MM} and C_{TSR-MM} are estimates of the coefficients based on TSR-MM estimator. The usual method in replicated responses problem is to firstly compute the sample mean and sample standard deviation of Y and construct the process mean and process standard deviation functions. Once the fitted response function for the process mean and process variance have been established, the optimum operating conditions of control factors are obtained by minimizing the following

$$\text{minimize } MSE = \hat{\sigma}^2(x) + (\hat{\mu}(x) - t_0)^2$$

where t_0 is the customer-identified target value for the quality characteristics of interest.

III. Results and Discussion

A. Monte Carlo Simulation

In this section, we report a Monte Carlo simulation study that is designed to assess the performance of the TSR-MM based estimator. In this simulation study, firstly, the responses Y were generated randomly from a normal distribution. Following (Park and Cho, 2003), five responses $(Y_{i1}, Y_{i2}, ..., Y_{i5})$ are generated from each distribution with $\mu(x_i)$ and $\sigma(x_i)$ at each control factor settings $x_i = (x_{i1}, x_{i2}, x_{i3})$ where $i = 1, 2, ..., 27$. The total number of iterations is 500, each having 27 design points, and 135 responses. $\mu(x)$ and $\sigma^2(x)$ are given as follows:

$$\mu(x) = 50 + 5(x_1{}^2 + x_2{}^2 + x_3{}^2)$$

$$\sigma^2(x) = 100 + 5((x_1 - 0.5)^2 + x_2{}^2 + x_3{}^2)$$

Secondly, to see how the lack of a normal distribution affects the estimators, the responses Y

are also generated from other distribution such as double exponential distribution, which has heavier tails distribution that is prone to produce a few outliers. The probability density functions is given by

$$f(x) = \frac{1}{2\beta} e^{-|x-\mu|/\beta}$$

In order to induce heteroscedasticity of the error variances, $\sigma^2(x)$ is generated according to this relation, $\sigma^2(x) \times exp(0.8{x_1}^2 + 0.8{x_2}^2 + 0.8{x_3}^2)$. To further investigate the effect of outliers, the data were contaminated by generating three outliers from $N(250, 10^2)$, that is 2.2% contamination. The good data were then replaced by 3 outliers such that the first observation of the second response variable, the 27th of observation of the third response variables, and the 14th observation of the fourth response variable are replaced with the outliers already generated. Since the OLS model is known to be not reliable in the presence of outliers, it is not included in the comparison. For each distribution specified above, two statistical measures such as bias and mean squared error (MSE) using RLS and TSR-MM based methods were considered as decision criteria to judge the performance of the estimators. Using the Breusch-Pagan test for non-constancy of error variance with the level of significance $\alpha = 0.05$, test statistics for testing $H_0 : \gamma_1 = \gamma_2 = \gamma_3 = 0$ versus $H_1 : \textit{not all } \gamma_i \textit{ equals } 0$ is calculated. Since test statistics $\chi^2_{BP} = 7.987 > \chi^2_{0.95,3} = 7.81$, H_0 is rejected, which suggests that the error variance of this study is not constant.

Table 1 (see Appendix) illustrates the estimated bias and MSE of the optimal mean response $\hat{\mu}(x)$ for response surface model with heteroscedastic errors based on RLS and TSR-MM based methods. Assuming that the target value for this experiment is $t_0 = 50.0$. It can be observed that in the presence of heteroscedascity and without contaminated data, as expected, the RLS based estimates is slightly better than the TSR-MM based. However, for non-normal data having heteroscedastic errors, the TSR-MM based method is more efficient that the RLS based method, evidence by having smaller bias and MSE.

B. Numerical Results

In this section, the performance of the newly proposed robust TSR-MM based estimator is assessed through two numerical examples.

1. Printing Process Data

This experiment introduced by (Box and R., 1987), was conducted to determine the effect of the three control variables: x_1(speed), x_2(pressure), and x_3(distance) on the characteristic of a printing process y, that is on the machine's index to apply colored inks to package labels . The experiment is a 3^3 factorial design with three replicates at each of the 27 design points. Table 2 (see Appendix) displays the data for this experiment, along with the calculations for the mean and standard deviation at each design point. In order to see the effect of outliers in the heteroscedasticity data, we deliberately changed three response points, that is the 8^{th}, 15^{th}, and 27^{th} observation corresponding to y_1(259 to 9259), y_2(568 to 8656), and y_3(1161 to 11161). The graphical results for the assumptions of normality and constant variance for original data are displayed in Figure 1. It can be seen from this figure that there is a moderate heteroscedasticity problems. Following (Goethals and Cho, 2011), based on level of significance, $\alpha = 0.05$ and assuming that $\log_e {\sigma_i}^2 = \gamma_0 + \gamma_1 X_{i1} + \gamma_2 X_{i2} + \gamma_3 X_{i3}$, the Breusch-Pagan test for non-constancy of error variance is conducted. The hypothesis test is stated as $H_0 : \gamma_1 = \gamma_2 = \gamma_3 = 0$ versus $H_1 : \textit{not all } \gamma_i \neq 0$ and since the value of Breusch-Pagan test statistics for a full second-order model is $\chi^2_{BP} = 37.219 > \chi^2_{0.95,3} = 7.81$, H_0 is rejected. Hence, the error variances of this experiment are not constant. The optimum response based on least-squares (OLS), Reweighted Least Squares (RLS based), and Two-Stage Robust (TSR-MM based) estimations were then applied to the data.

The full second-order response surface

functions for the process mean and process standard deviation based on the OLS, RLS and TSR-MM based respectively, are given as follows.

(1) OLS

$$\hat{\sigma}_{OLS}(x) = 328.12 + 177.0x_1 + 109.43x_2 + 131.28x_3 + 31.63{x_1}^2 - 22.76{x_2}^2 - 28.87{x_3}^2 + 66.03x_1x_2 + 75.47x_1x_3 + 43.58x_2x_3$$

$$\hat{\sigma}_{OLS}(x) = 35.29 + 11.53x_1 + 15.32x_2 + 29.04x_3 + 3.90{x_1}^2 - 1.62{x_2}^2 - 16.93{x_3}^2 + 7.72x_1x_2 + 5.11x_1x_3 + 14.08x_2x_3$$

(2) RLS

$$\hat{\mu}_{RLS}(x) = 314.21 + 174.32x_1 + 136.49x_2 + 127.17x_3 + 28.46{x_1}^2 - 8.563{x_2}^2 - 13.95{x_3}^2 + 37.49x_1x_2 + 47.76x_1x_3 + 67.12x_2x_3$$

$$\hat{\sigma}_{RLS}(x) = 44.708 + 22.654x_1 + 25.787x_2 + 33.429x_3 - 6.773{x_1}^2 - 19.797{x_2}^2 + 22.485{x_3}^2 - 19.458x_1x_2 + 22.805x_1x_3 + 25.777x_2x_3$$

(3) TSR-MM based

$$\hat{\mu}_{TSR-MM}(x) = 285.43 + 178.82x_1 + 135.98x_2 + 119.14x_3 + 53.13{x_1}^2 + 19.30{x_2}^2 - 10.91{x_3}^2 + 58.20x_1x_2 + 51.42x_1x_3 + 51.53x_2x_3$$

$$\hat{\sigma}_{TSR-MM}(x) = 39.93 + 17.88x_1 + 25.11x_2 + 21.54x_3 + 3.04{x_1}^2 + 2.72{x_2}^2 - 2.04{x_3}^2 - 4.86x_1x_2 + 14.04x_1x_3 + 24.01x_2x_3$$

Table 3 (see Appendix) presents the standard errors of each estimate for the process mean and process standard deviation using the OLS, RLS, and TSR-MM based. Without the presence of outliers in the data, the standard errors of each of the three estimates are small and fairly closed to each other. Nonetheless, the TSR-MM based estimator is slightly better than the RLS and the OLS as evidenced by its smallest standard errors for both mean and standard deviation response model. Table 4 (see Appendix) exhibits the estimated optimum settings, mean, variance, and MSE of the estimated mean response. The mean squared error is obtained by the MSE relation where, $MSE = \hat{\sigma}^2(x) + (\hat{\mu}(x) - t_0)^2$ with $t_0 = 500$. It can be seen from Table 4 that the estimated mean response based on RLS achieves the target i.e. 500 and has the smallest value of MSE.

It can be observed from Figure 2, in the presence of outliers, the plots revealed a serious problem of non-constant variance and non-normality. The results of Table 5 and Table 6 (see Appendix) signify that in the presence of outliers, things change dramatically. The OLS and RLS based are immediately affected by outliers. It can be seen that the standard errors of the OLS and RLS estimates increased markedly, and their objective target have deviated. Nevertheless, as expected, the TSR-MM based estimate is only slightly affected by outliers revealed by smaller values of the standard errors, and MSE and achieve the objective target.

IV. Conclusion

The main aim of this chapter was to develop a reliable alternative approach in dual response model for correcting the problem of heteroscedastic errors in the presence of outliers. It seems that the performances of the optimum mean response of the RLS and the TSR-MM based estimators are equally good in a heteroscedascity data without outliers. The RLS based estimator is a good technique for solving heteroscedascity problem but it is easily affected by outliers. Hence, they are not reliable. In this chapter, we proposed a TSR-MM based method where it can remedy both problems of heteroscedascity and outliers at the same time. The numerical example and simulation experiment indicate that the TSR-MM based method offers a substantial improvement over the other existing methods for handling the problems of outliers and heteroscedastic errors in response surface model.

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Appendix A Figures and Tables

Table 1: Estimated Bias and MSE of the Estimated Optimal Mean Response for Heteroscedascity Data Using RLS based and TSR-MM based Methods

	RLS based		TSR-MM based	
Distribution	**Bias**	**MSE**	**Bias**	**MSE**
Normal	3.83	24.90	3.90	26.12
Normal (contaminated)	9.17	133.48	3.37	19.55
Double Exponential	4.65	40.95	4.15	29.54

Table 2: Printing Process Data Set

Run	**Speed,** X_1	**Pressure,** X_2	**Distance,** X_3	y_1	y_2	y_3	$\bar{y}$	s
1	-1	-1	-1	34	10	28	24.00	12.490
2	0	-1	-1	115	116	130	120.33	8.386
3	1	-1	-1	192	186	263	213.67	42.829
4	-1	0	-1	82	88	88	86.00	3.464
5	0	0	-1	44	188	188	140.00	83.138
6	1	0	-1	322	350	350	340.67	16.166
7	-1	1	-1	141	110	86	112.33	27.574
8	0	1	-1	259	251	259	256.33	4.619
9	1	1	-1	290	280	245	271.67	23.629
10	-1	-1	0	81	81	81	81.00	0.000
11	0	-1	0	90	122	93	101.67	17.673
12	1	-1	0	319	376	376	357.00	32.909
13	-1	0	0	180	180	154	171.33	15.011
14	0	0	0	372	372	372	372.00	0.000
15	1	0	0	541	568	396	501.67	92.500
16	-1	1	0	288	192	312	264.00	63.498
17	0	1	0	432	336	513	427.00	88.606
18	1	1	0	713	725	754	730.67	21.079
19	-1	-1	1	364	99	199	220.67	133.822
20	0	-1	1	232	221	266	239.67	23.459
21	1	-1	1	408	415	443	422.00	18.520
22	-1	0	1	182	233	182	199.00	29.445
23	0	0	1	507	515	434	485.33	44.636
24	1	0	1	846	535	640	673.67	158.210
25	-1	1	1	236	126	168	176.67	55.510
26	0	1	1	660	440	403	501.00	138.935
27	1	1	1	878	991	1161	1010.00	142.454

Table 3: Comparison of Mean and Standard Deviation of the Estimates for Original Dataset

Model	$s(b_0)$	$s(b_1)$	$s(b_2)$	$s(b_3)$	$s(b_4)$	$s(b_5)$	$s(b_6)$	$s(b_7)$	$s(b_8)$	$s(b_9)$
Mean-OLS	38.72	17.92	17.92	17.92	31.04	31.04	31.04	21.95	21.95	21.95
Mean-RLS	23.82	13.56	14.12	15.44	21.75	19.78	22.87	18.11	20.08	18.85
Mean-TRS-MM	29.57	10.07	11.81	12.92	25.31	15.24	15.96	10.79	13.74	14.44
Std-OLS	22.43	10.38	10.38	10.38	17.98	17.98	17.98	12.71	12.71	12.71
Std-RLS	18.18	9.29	9.52	9.98	14.91	13.48	13.55	11.54	11.57	13.00
Std-TRS-MM	15.94	6.24	5.78	6.39	10.26	11.54	9.64	5.73	7.37	6.73

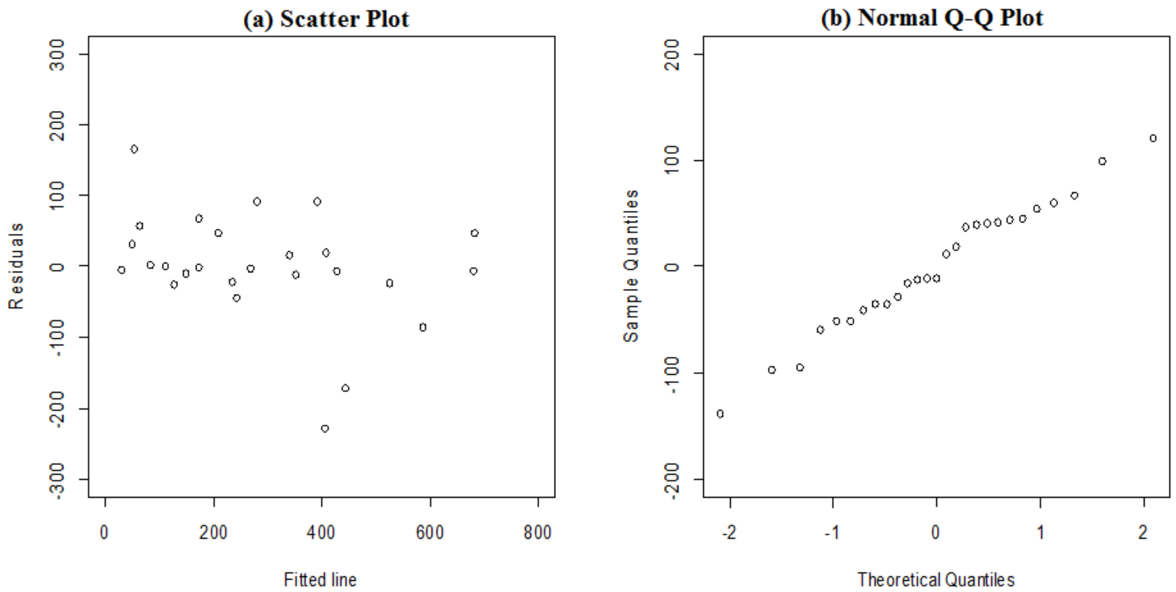

Figure 1: (a) Plot residuals against fitted line, and (b) Plot of normality for Original data set

Table 4: The Estimated Optimum Settings, Mean, Variance, and MSE of the Estimated Mean Response

Model	$\mathbf{x}^*$	Mean	Variance	MSE
OLS	$(1.000, 0.060, -0.243)$	494.657	1988.550	2017.099
RLS	$(0.9966, 0.9967, -0.7190)$	500	$8.043e^{-11}$	$5.161e^{-10}$
TRS-MM	$(1.000, 1.000, -1.000)$	497.86	492.29	496.85

Table 5: Comparison of Mean and Standard Deviation Models for Modified Dataset

Model	$s(b_0)$	$s(b_1)$	$s(b_2)$	$s(b_3)$	$s(b_4)$	$s(b_5)$	$s(b_6)$	$s(b_7)$	$s(b_8)$	$s(b_9)$
Mean-OLS	539.1	249.5	249.5	249.5	432.2	432.2	432.2	305.6	305.6	305.6
Mean-RLS	226.5	149.7	151.1	140.0	139.6	201.8	110.9	149.8	114.4	140.9
Mean-TRS-MM	33.21	15.72	18.13	16.94	31.31	24.17	23.79	20.17	18.35	21.13
Std-OLS	912.5	422.4	422.4	422.4	731.6	731.6	731.6	517.3	517.3	517.3
Std-RLS	360.7	264.2	265.5	245.7	236.5	331.9	202.1	268.4	236.7	253.7
Std-TRS-MM	18.24	6.33	8.19	7.00	13.06	11.90	11.89	8.05	7.52	9.92

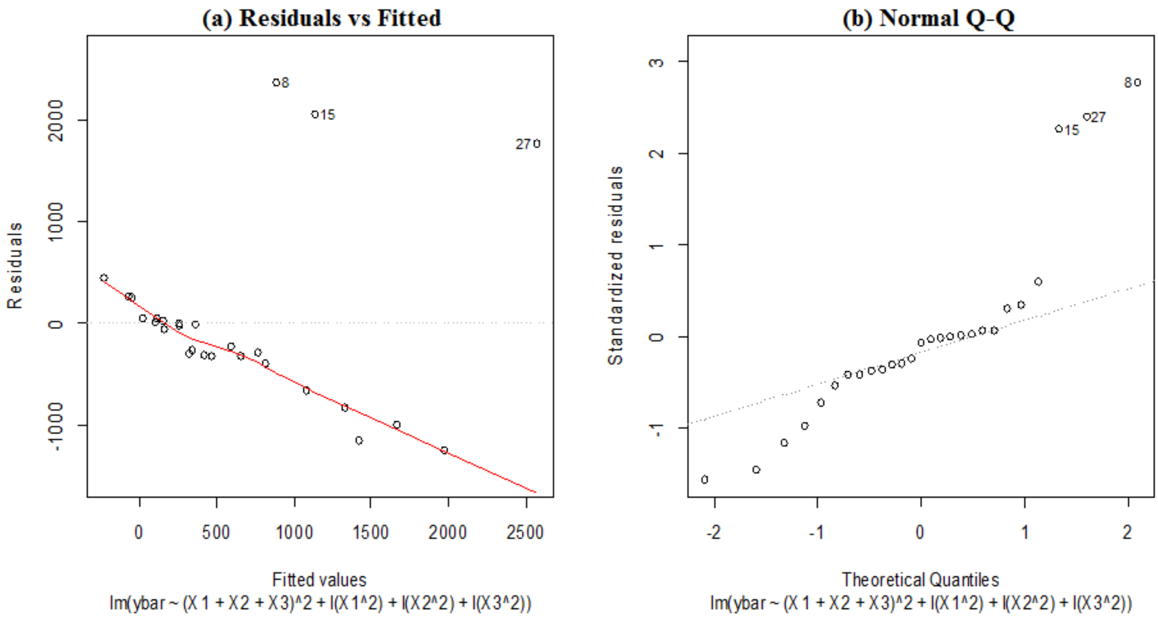

Figure 2: (a) Plot residuals against fitted line, and (b) Plot of normality for Modified data set

Table 6: The Estimated Optimum Settings, Mean, Variance, and MSE of the Estimated Mean Response for Modified dataset

Model	$\mathbf{x}^*$	**Mean**	**Variance**	**MSE**
OLS	$(-0.637, 0.353, 1.000)$	342.01	14448.21	39407.22
RLS	$(0.777, -1.000, 1.000)$	444.97	12482.95	15511.21
TRS-MM	$(1.000, 0.1278, -0.3421)$	497.62	793.13	798.81