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Construction of Quintic Trigonometric Bézier Spiral Curve

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The construction of curves need to satisfy some conditions in order to achieve smooth pathways. In terms of designing highways, there are five design templates of transition curves. Usually, cubic and quintic Bézier curves are used to construct those five templates. However, quintic trigonometric Bézier curves can also be used to construct these five templates of transition curves. Global and local dynamic parameters that are embedded in control points of quintic trigonometric Bézier will provide more freedom for designers to comply with difficult tasks in designing.

Keywords: spiral; smooth pathway; quintic curve; trigonometric Bézier; dynamic parameters

I. INTRODUCTION

Spiral curves can be found from the representation of growth and the process of the Nautilus shell, to the patterns of sunflower seeds, to the design of a staircase in the Vatican Museum, and to the method of construction of railway/roadway design. A spiral curve is one of the features in geometric field especially in Computer Aided Geometric Design. This important feature can be added on to a regular circular curve. The spiral can be defined as a curve segment that provides a gradual transition between a straight line to a circle. In engineering, spirals are used to overcome the abrupt change in curvature and superelevation that occurs between tangent and circular curve especially for moving objects (Misro *et al.*, 2015).

The study of the natural structures will help numerous designers, architects, or even engineers to learn the secrets of each natural form. Archetypal motif in spiral curve itself brings its own uniqueness that can be represented in various forms. Spirals that consist of strength, versatility, and pleasing looks will be the indicator for designers. The study of the natural spiral form is inclusive of the features of human face that were derived by using golden spiral, whereby the ratio represented by the spiral constant is the only growth ratio by which a new unit can grow in proportion to the old unit; and still retain the same shape. This is exactly the same process of growth in a spiral shell.

Spiral curves consist of several types beginning with Euler Spiral, Logarithmic Spiral, Parabolic Spiral, and more. Euler spiral is a beautiful and useful curve known by several other names, including clothoid and Cornu spiral (Levien, 2008). One unique property of this spiral curve is that the curvature increases linearly with its arclength. Levien (2018) stated that the problem of elasticity were the first appearance of Euler Spiral published by James Bernoulli in 1694.

The logarithmic, or equiangular spiral was first discovered by Rene Descartes (1596-1650) in 1637 by representing various curves with the help of equations. Logarithmic spiral was one of those curves which at that time drew the attention of mathematicians. At present, that equation is written in the form $r = e^{\alpha\theta}$. A parabolic spiral can also be represented by the mathematical equation $r^2 = a^2\theta$. This spiral discovered by Bonaventura Cavalieri (1598-1647) creates a curve commonly known as a parabolic spiral (Mukhopadhyay, 2004).

Recently, the study of spiral construction using any kind of method has aspired many researchers, including the use of trigonometric Bézier curve that comprises the flexibility of dynamic parameters. Previously, (Misro *et al.*, 2017) had published his work about spiral curve using cubic trigonometric Bézier curve. This research proved that spiral curves can be constructed using trigonometric function and not limited to only polynomial and hyperbolic function (logarithmic spiral).

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In this research, a method to construct a spiral curve by increasing the number of segments will be proposed. Therefore, a higher degree of curve is required in order to bring more flexibility to the spiral curve. Section 2 briefly explains the background notations used throughout this research including the basis function of quintic trigonometric Bézier curve. Section 3 summarises all the conditions of the Pythagorean Hodograph (PH) quintic trigonometric Bézier spiral curve. Another four templates of transition curves will be demonstrated in Section 4 to 7. Some remarks and suggestions for future works are concluded in Section 8.

II. NOTATION AND CONVENTION

Quintic Trigonometric Bézier curve with local dynamic parameters p and q proposed by (Misro *et al.*, 2017b) will be used throughout this paper, and given as follows

$$z\,(t) = \sum_{i=0}^{5} P_i g_i(t) \tag{1}$$

where P_i are control points and g_i be the quintic trigonometric basis function for every $i = 0, 1, 2, 3, 4, 5$ which are defined as

$$f_0(t) = (1 - sin\tfrac{\pi t}{2})^4 \left(1 - p\, sin\tfrac{\pi t}{2}\right),$$
$$f_1(t) = sin\tfrac{\pi t}{2}(1 - sin\tfrac{\pi t}{2})^3 \left(4 + p - p\, sin\tfrac{\pi t}{2}\right),$$
$$f_2(t) = \left(1 - sin\tfrac{\pi t}{2}\right)^2 \left(1 - cos\tfrac{\pi t}{2}\right)\left(8\, sin\tfrac{\pi t}{2} + 3\, cos\tfrac{\pi t}{2} + 9\right),$$
$$f_3(t) = \left(1 - cos\tfrac{\pi t}{2}\right)^2 \left(1 - sin\tfrac{\pi t}{2}\right)\left(8\, cos\tfrac{\pi t}{2} + 3\, sin\tfrac{\pi t}{2} + 9\right),$$
$$f_4(t) = cos\tfrac{\pi t}{2}(1 - cos\tfrac{\pi t}{2})^3 \left(4 + q - q\, cos\tfrac{\pi t}{2}\right),$$
$$f_5(t) = (1 - cos\tfrac{\pi t}{2})^4 (1 - q\, cos\tfrac{\pi t}{2}). \tag{2}$$

Local dynamic parameters of p and q in Equation (2) will enable the curve to become more flexible. The curve $z(t) = (x(t), y(t))$ is said to be a PH curve if $x'(t)^2 + y'(t)^2$ can be expressed as the square of polynomial in t. By using the result given by (Farouki & Sakkalis, 1990), control points P_i of the quintic trigonometric Bézier curve are as follows

$$P_1 = P_0 + \frac{1}{5}(u_0^2 - v_0^2, 2u_0v_0),$$
$$P_2 = P_1 + \frac{1}{5}(u_0u_1 - v_0v_1, u_0v_1 + u_1v_0),$$
$$P_3 = P_2 + \frac{1}{15}(2u_1^2 - 2v_1^2 + u_0u_2 - v_0v_2, 4u_1v_1 + u_0v_2 + u_2v_0),$$
$$P_4 = P_3 + \frac{1}{5}(u_1u_2 - v_1v_2, u_1v_2 + u_2v_1),$$
$$P_5 = P_4 + \tfrac{1}{5}(u_2^2 - v_2^2, 2u_2v_2), \tag{3}$$

where P_0 is arbitrarily chosen. PH curve can be defined in parametric form as

$$z'(t) = \left(u(t)^2 - v(t)^2, 2u(t)v(t)\right),\ \ 0 \le t \le 1 \tag{4}$$

Where

$$u(t) = u_0(1-t)^2 + 2u_1(1-t)t + u_2t^2,$$
$$v(t) = v_0(1-t)^2 + 2v_1(1-t)t + v_2t^2. \tag{5}$$

Its signed curvature $\kappa(t)$ is given by

$$\kappa(t) = \frac{z'(t) \times z''(t)}{\|z'(t)\|^3} = \frac{2\left(u(t)v'(t) - u'(t)v(t)\right)}{(u(t)^2 + v(t)^2)^2} \tag{6}$$

III. QUINTIC TRIGONOMETRIC BÉZIER SPIRAL CURVE

Apart from cubic Bézier curve, PH quintic Bézier curve is oftenly used as a basis function to construct transition curve (Habib & Sakai, 2005; Walton & Meek, 2004). In (Misro *et al.*, 2017c), the authors only constructed PH quintic trigonometric Bézier transition curve. Therefore, in this paper, we will construct spiral curves using quintic trigonometric Bézier with the following conditions to be satisfied

$$z(0) = (0,0),\ \ \left(\kappa(0), \kappa(1)\right) = \left(0, \tfrac{1}{r}\right), \tag{7}$$

$$z'(0) \parallel (1,0),\ \ z'(1) \parallel (\cos\theta, \sin\theta) \tag{8}$$

$\theta \in (0, \frac{\pi}{2})$ is an angle that is measured anti-clockwise from $z'(0)$ to $z'(1)$. Parameter r is referred as the radius of the circle. Taking quintic trigonometric Bézier curve in Equation (1) with basis function in Equation (2), our initial starting point will be at $z(0) = (0, 0)$ in Equation (7). To have a spiral curve from a straight line to a circle, we need $\kappa(0) = 0$ and $\kappa(1) = \frac{1}{r}$. Its curvature begins with zero at the straight line and increases linearly with its curve length. Both initial value and end value of the curvature are crucial for this spiral curve to ensure a smooth continuity at the joints. Then, parallel equation in Equation (8) needs to be satisfied before we can obtain the following unknowns such as

$$(u_0, u_1, u_2) = \left(m\frac{d^3}{4r\sin\frac{\theta}{2}}, \frac{d^3}{4r\sin\frac{\theta}{2}}, d\cos\frac{\theta}{2} \right),$$
$$(v_0, v_1, v_2) = (0,0,d\sin\tfrac{\theta}{2}) \quad (9)$$

with positive parameters $u_0 = mu_1$ and $d = \sqrt{sr}$ whereas m and s are free parameters to alter the shape of the curve. The parameter m and u_0 act as global dynamic parameters that will depend on the value of θ. Thus, quintic spiral curves $z(t)$ that are satisfying the conditions in Equation (7) and Equation (8) can be found. Therefore, the quintic trigonometric Bézier spiral curve can be written a

$$z(t) = \sum_{i=0}^{5} P_i g_i(t) = P_0 g_0(t) + P_1 g_1(t) + P_2 g_2(t) + P_3 g_3(t) + P_4 g_4(t) + P_5 g_5(t) \quad (10)$$

with $P_i = 1, 2, 3, 4, 5$ as in Equation (3).

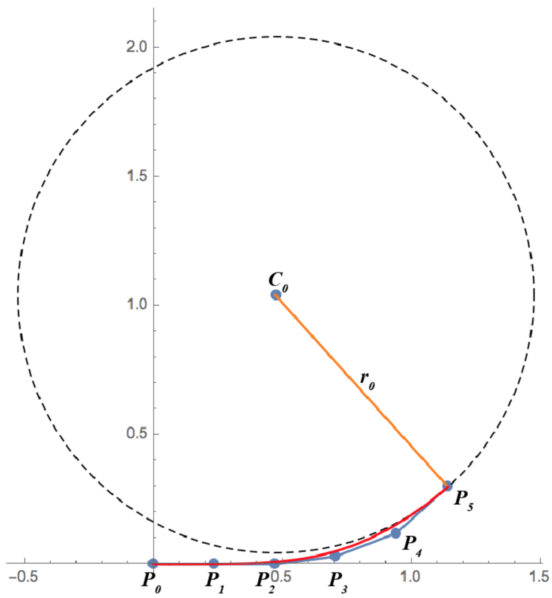

Figure 1. A spiral quintic trigonometric Bézier curve

Figure 1 shows a spiral curve that is constructed using quintic trigonometric Bézier curve. In Figure 1, the radius of circle is $r_0 = 1$ with $\theta = 42$ degree and $s = \frac{7}{2}\tan\frac{\theta}{2}$. The value of global parameter is $m = 1$ with local dynamic parameters $p = 0$ and $q = 0$. The local parameter is said to be dynamic because it varies from -4 to 1.

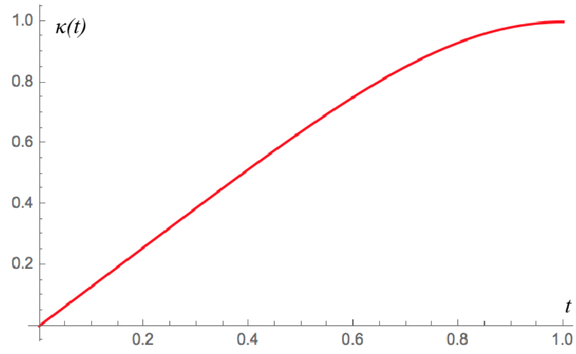

Figure 2. Curvature profile of spiral quintic trigonometric Bézier curve

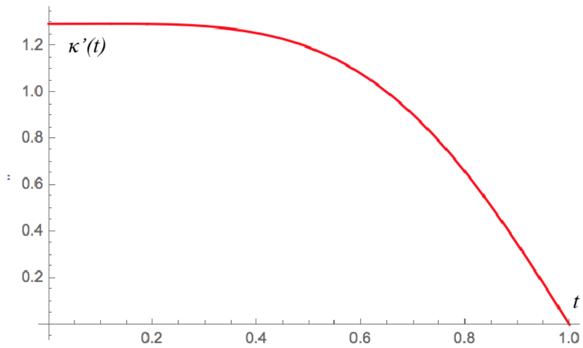

Figure 3. Curvature derivative of spiral quintic trigonometric Bézier curve

Curvature profile in Figure 2 is almost linearly increasing until $t = 0.85$, where it starts to have a flat value until $t = 1$. This shows that the curve is entering the circular curve of the circle at the end of the curvature. Validation process is made by providing curvature derivative distributions of this spiral curve in Figure 3 to ensure that it has only one sign from $t = 0$ to $t = 1$.

For Sections 4, 5, and 6, we will used two pieces of quintic trigonometric Bézier spiral curves that were developed earlier to construct another three templates. Therefore, the general equation for two pieces of spiral curves can be defined as follows

$$z_1(t) = \sum_{i=0}^{5} P_i g_i(t), \quad z_2(t) = \sum_{i=0}^{5} B_i g_i(t) \quad (11)$$

where its respective curvatures are

$$\kappa_1(t) = \frac{z'(t) \times z''(t)}{\|z'(t)\|^3}, \quad \kappa_2(t) = \frac{z'(t) \times z''(t)}{\|z'(t)\|^3}. \quad (12)$$

IV. STRAIGHT LINE TO

STRAIGHT LINE SPIRAL CURVE

In this section, the spiral curve will be extended into a curve from a straight line to a straight line. Given two pieces of quintic trigonometric Bézier spiral curves as in Equation (11), the curve can be joined as a straight line to straight line spiral curve when its curvatures in Equation (12) satisfies the G^2 Hermite condition below

$$\kappa_1(1) = \frac{1}{r} = \kappa_2(0) \qquad (13)$$

with r as the radius of the circle that connects both curves at the last control point of the first curve and the first control point of the second curve.

Using the same basis function in Equation (2) for two pieces of quintic trigonometric Bézier curve in Equation (11), the first piece of quintic trigonometric Bézier spiral segment follows the same method as in previous section where the curvature value will end at $\kappa_1(1) = \frac{1}{r}$. Thus, the equation of the spiral is the same as in Equation (10).

For the second piece of quintic trigonometric Bézier spiral, this segment must be connected at the same point, tangent, and curvature value to ensure a smooth continuity between two straight lines. Therefore, the initial curvature value for the second curve must be the same with the end curvature value of the first curve which is $\frac{1}{r} = \kappa_2(0)$. Hence, the curvature value at the common point between two straight lines is $\kappa_1(1) = \frac{1}{r} = \kappa_2(0)$, as stated in Equation (13). Therefore, $P_5 = B_0$ will be the common point or the point of contact for this curve as in Figure 4.

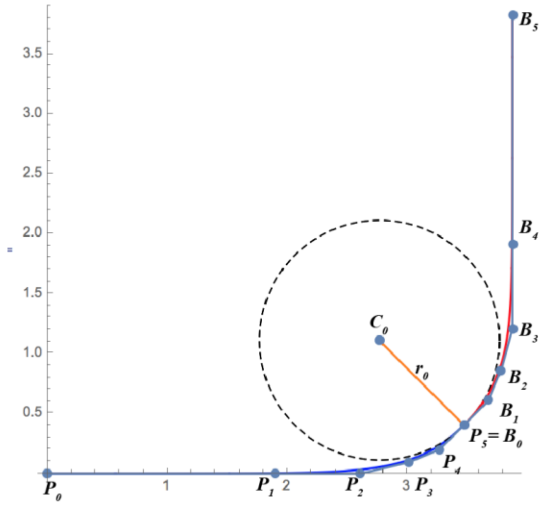

Figure 4. Straight line to straight line curves with control points

Figure 4 shows a straight line to a straight line curve that was constructed using two spiral segments of quintic trigonometric Bézier curve at a specific point of contact with radius $r = 2$, turning angle $\theta = 45$ degree, $s = s_1 = s_2 = \frac{7}{2}\tan\frac{\theta}{2}$ and value of dynamic parameter $p = 1, q = 1$. Figure 5 shows the same curve that was constructed without its control points.

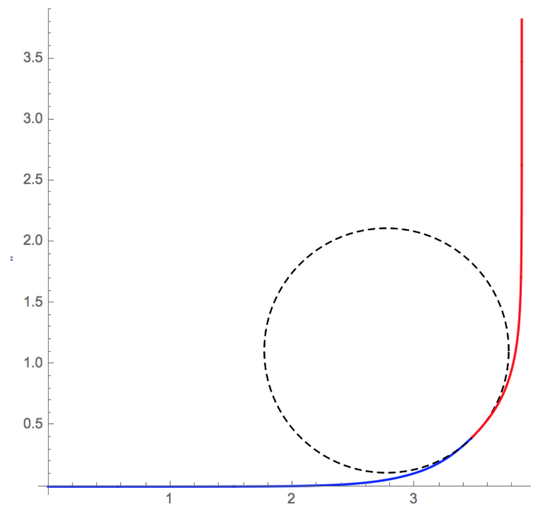

Figure 5. Straight line to straight line composed of two spiral curves

Figure 4 shows two pieces of spiral curve segments that meet at the same curvature value which is at the points $P_5 = B_0$. The straight line to straight line curve that is constructed from quintic trigonometric Bézier spiral curve is useful not only for designing highways, but it can also be used for rounding objects such as canoe, furniture, paddles, and so on.

V. S-SHAPED SPIRAL CURVE

An S-shaped curve is a transition curve that is composed by two spiral curves at a specific point of contact. Given two circle ω_0 and $\omega_1$with radii r_0 and r_1 respectively. Let the first quintic trigonometric Bézier curve denoted as $z_0(t) = (x_0(t), y_0(t))$be the first segment of quintic trigonometric Bézier spiral curve with the same method as in Section 3.

Then, the second segment of quintic trigonometric Bézier spiral curve with the same definition of control points are denoted as $z_1(t) = \left(-x_1(t), -y_1(t)\right)$. S-shaped curve composed of two spiral segments can be formed when $\| c_0 - c_1 \| > \| r_0 + r_1 \|$ and both spiral segments of curve must meet at the same point of contact, which in this case have zero curvatures but different in $x(t)$ and $y(t)$ directions. Construction of S-shaped curve composed by two quintic trigonometric Bézier spiral curves can be done using Equation (11) with its respective curvatures in Equation (12) agrees the following condition

$$\kappa_0(0) = 0 = \kappa_1(0) \quad (14)$$

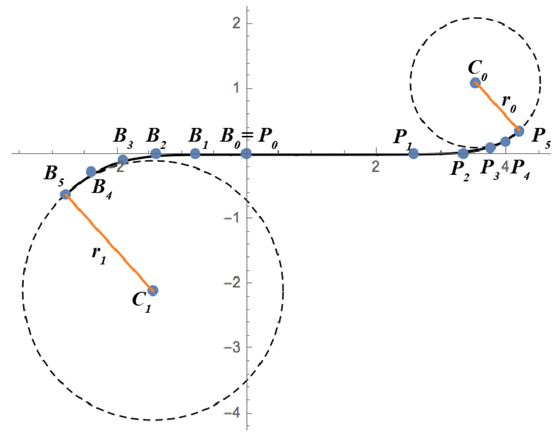

Figure 6. S-shaped spiral curve with $(m_0, m_1) = (3.293, 1.293)$

The S-shaped curve can be constructed by using two quintic trigonometric Bézier spiral curves as shown in Figure 6 with $\theta = 42$, value of dynamic parameters $p = 1,\ q = 1$, radii of the circles are $r_0 = 1, r_1 = 2$ and the value for $m \geq \frac{2}{7}(\sqrt{30} - 3)$ as stated by (Habib & Sakai, 2005). Therefore, the selection of value m must satisfy the condition of the spiral curve. This S-shaped curve is made up of two spiral segments of quintic trigonometric Bézier curves. Therefore, there must be two control points that joined two curves at the same point of contact. In this case, B_0 meets P_0 with the curvature value as in Equation (14).

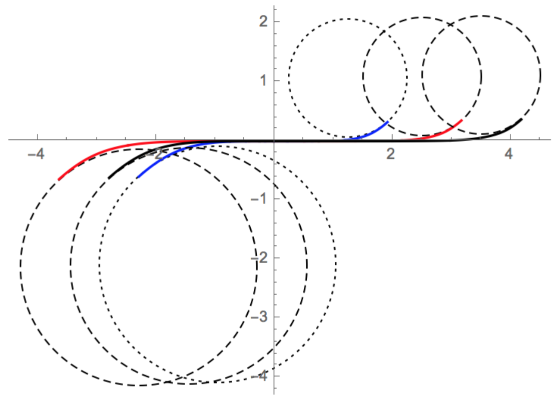

Figure 7. S-shaped spiral curve with $S_{red} = (m_0, m_1) = (2.707, 1.707)$, $S_{blue} = (m_0, m_1) = (1.8, 1.0)$, and $S_{black} = (m_0, m_1) = (3.293, 1.293)$

Figure 7 shows S-shaped curves that are composed by two spiral segments using different parameters m but with the same local dynamic parameter $p = 1, q = 1$. Different values of global parameter m in this figure will determine the length between the first control point to the second control point on each particular spiral segment. Red curve in Figure 7 used two pairs of parameter m which are $m_0 = 2.707$ and $m_1 = 1.707$. The blue curve used $(m_0, m_1) = (1.8, 1)$ and the black curve used $(m_0, m_1) = (3.293, 1.293)$.

VI. C-SHAPED SPIRAL CURVE

Similar to S-shaped curve, C-shaped curve is also a transition curve that is composed by two spiral curves at a specific point of contact. ω_0 and ω_1 are two circles with radii r_0 and r_1 respectively. The first spiral segment similar to S-shaped curve that is denoted as $z_0(t) = \left(x_0(t), y_0(t)\right)$. with all the condition in Equations (7) and (8) are satisfied.

Let $z_1(t) = \left(-x_1(t), y_1(t)\right)$ to be another segment of quintic trigonometric spiral with the analogous sets of control points. C-shaped curve composed by two spiral segments can be formed when $\| r_0 + r_1 \| > \| c_1 - c_1 \|$ and both spiral segments of curve meet at the same point of contact which have zero curvatures but both differ in its direction like in Section 5. Therefore, C-shaped transition curve composed by two quintic trigonometric Bézier spirals as in Equation (11) can be constructed when Equation (12) satisfy the same condition as in Equation (14).

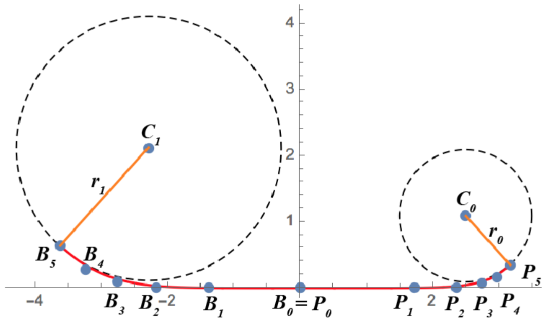

Figure 8. C-shaped spiral curve with $(m_0, m_1) = (2.707, 1.707)$

A C-shaped transition curve can be constructed by using two quintic trigonometric Bézier spiral curves as shown in Figure 8. This C-shaped curve is also made up of two spirals of quintic trigonometric Bézier curve by joining at the same points of contact at $P_0 = B_0$ with the same curvature value which is $\kappa_1(0) = 0 = \kappa_0(0)$ as in Equation (14).

Different values of parameter m in Figure 9 will determine the length between the first control point and the second control point on each particular spiral segment. Each color of C-shaped curve using two different values of parameter m which are $(m_0, m_1) = (2.707, 1.707)$ for red curve, $(m_0, m_1) = (1.8, 1)$ for the blue curve and $(m_0, m_1) = (3.293, 1.293)$ for the black curve as shown in Figure 9.

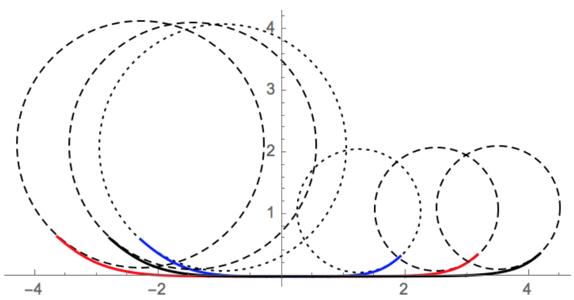

Figure 9. C-shaped spiral curve with $C_{red} = (m_0, m_1) = (2.707, 1.707)$, $C_{blue} = (m_0, m_1) = (1.8, 1.0)$, and $C_{black} = (m_0, m_1) = (3.293, 1.293)$

VII. CIRCLE IN CIRCLE SPIRAL CURVE

In this section, circle in circle C-shaped curve can also be constructed using quintic trigonometric Bézier curve. Given two circles, ω_0 and ω_1 with radii r_0 and r_1 accordingly and such that ω_1 is the smaller circle that is contained completely inside the bigger circle ω_0. Let the quintic trigonometric Bézier curve denoted as $z_0(t) = (x_0(t), y_0(t))$ be the curve segment with basis function in Equation (2). To construct a quintic trigonometric Bézier spiral curve, the condition in Equation (7) and Equation (8) need to be satisfied along with the same control points as mentioned in Equation (3).

For example, C-shaped spiral curve of quintic trigonometric Bézier curve in Figure 10 is constructed using $p = 0$, $q = 0$ with $r_0 = 4$, $r_1 = 1$ respectively. The centre for big circle is $c_0 = (0.32, 4)$ while small circle is $c_1 = z_0(1) + r_1(-\sin\theta, \cos\theta)$ with $\theta = 52$ degree. Then, the value of parameter $m = 1$ and $s = \frac{7}{2}\tan\frac{\theta}{2}$. The process to obtain a circle in circle C-shaped spiral is similar to the process to obtain a straight line to a circle as in Section 3 but we need to be careful in order to choose the initial control point that are also attached to another circle. Hence, a circle in circle curve can be constructed with a single quintic trigonometric Bézier spiral curve.

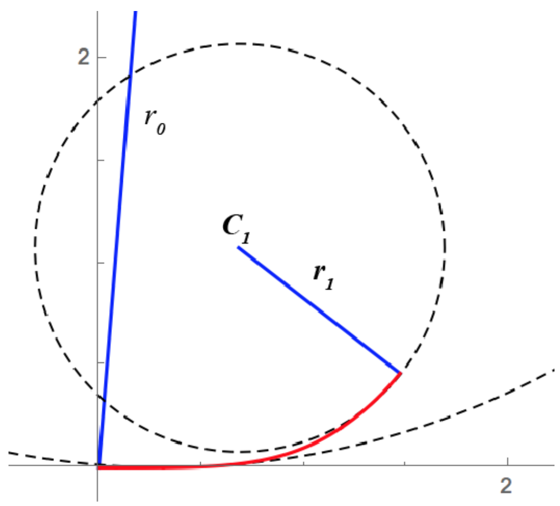

Figure 10. Circle in circle spiral curve

Figure 11 shows the curvature profile of quintic trigonometric Bézier spiral curve for a circle in circle spiral curve. Curvature profile for this C-shaped spiral curve of quintic trigonometric Bézier is monotonically increasing. Figure 12 depicts its corresponding curvature derivative. The signs of curvature derivative does not change along intervals $t \in [0,1]$.

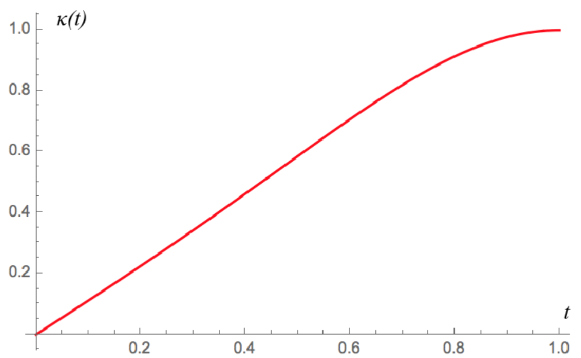

Figure 11. Curvature profile of circle in circle quintic trigonometric Bézier spiral curve

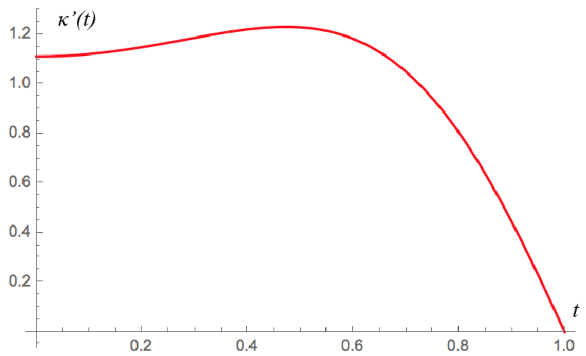

Figure 12. Curvature derivative of circle in circle quintic trigonometric Bézier spiral curve

VIII. CONCLUSION

Construction of spiral curves need to satisfy some conditions in order to achieve smooth pathways. In terms of designing highways, there are five design templates of transition curves (Misro *et al.*, 2018; Misro *et al.*, 2017d) Usually, cubic Bézier curves are used to construct those five templates. However, quintic Bézier curves can also be used to construct these five templates of transition curve. In (Misro *et al.*, 2017e), the construction of five templates of transition curve using cubic trigonometric Bézier spiral curve was done, while in this paper, the construction is made by using quintic trigonometric Bézier spiral curve.

Inflection points will always occur during the construction of S-shaped transition curve, whereas curvature extrema will exist on C-shaped transition curve. Therefore, two spiral curves are composed to construct a straight line to straight line transition curve and a circle to circle transition curve in order to avoid these drawbacks. However, other issues arise when two spiral curves are used, where a specific point of contact with the same points, tangents, and curvatures are needed. Therefore, by using the same points of contact between two segments by sharing the same tangents and curvatures will provide a precise geometrical interpretation. Thus, the two spiral curves are joined continuously and these spiral curves can be used to design highways or railways and to determine the maximum speed estimation (Ibrahim *et al.*, 2017).

IX. ACKNOWLEDGEMENTS

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