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High-order Compact Iterative Scheme for the Two-dimensional Time Fractional Cable Equation

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In this paper, we present a high-order compact scheme for the solution of the two-dimensional time fractional cable equation. The Caputo fractional derivative operator is used for the time derivative and a fourth-order compact Crank-Nicolson approximation is used for the space derivative to produce a high-order compact implicit scheme. The proposed method will be shown to have the order of convergence $O(\tau^{2-\alpha} + \mathrm{h}^4)$. Finally, to show the accuracy of the proposed scheme, some numerical examples are provided.

Keywords: Two-dimensional fractional cable equation; Crank Nicolson; High-order compact scheme; Finite difference method.

I. INTRODUCTION

In recent years, fractional Calculus has gained attention due to its applications in various fields of science and technology [1-5]. To solve fractional differential equations different numerical and analytical methods are proposed for example, Finite difference method, Finite element method, Finite volume method, Adomian Decomposition method [6-10] etc. In the numerical methods, Finite difference method has seen more in the literature for solving fractional differential equations [11-24].

The fractional cable equation is derived from the Nernst-Planck equation which gives us a macroscopic approximation of the complicated microscopic motions of ions in nerve cells [25]. Different numerical methods are proposed for solving fractional cable equation for example, Liu et al. [26] solved one dimensional fractional cable equation by two implicit numerical methods with second-order spatial accuracy, Chen et al. [27] solved one dimensional non-linear variable order fractional cable equation with fourth order spatial accuracy, Zhang et al. [28] solved two- dimensional fractional cable equation by discrete-time orthogonal spline collocation methods, Balasim and Ali [29] used implicit schemes for the solution of two-dimensional fractional cable equation with second-order of spatial accuracy, and Bhrawy and Zaky [30] solved one and two dimensional fractional cable equation by spectral collocation method. However, computationally effective high order implicit numerical methods for solving two-dimensional fractional cable equations are still in their infancy.

The purpose of this paper is to propose a compact high order numerical scheme for the solution of two-dimensional fractional cable equation, which is fourth-order accurate in space. The paper is organized as follows; formulations of the compact Crank Nicolson method is discussed in section 2, numerical examples and results are presented in section 3 and finally, the conclusion in section 4.

II. SCHEME FORMULATION

The two dimensional time fractional cable equation is

$$ {}_0^C D_t^\alpha u(x,y,t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \mu_0 u(x,y,t) + f(x,y,t), \qquad (1)$$

Where $(x,y) \in (\mathrm{L}_1,\mathrm{L}_2)\times(\mathrm{L}_3,\mathrm{L}_4)$, and $0 < t < T$.

Caputo fractional derivative is represented by ${}_0^C D_t^\alpha u$ $(0 < \alpha < 1)$, which is defined by [31],

$$ {}_0^C D_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(x,y,\tau)}{(t-\tau)^\alpha} \partial\tau, \quad 0 < \alpha < 1,$$

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where $\Gamma(.)$ represents gamma function and

$u'(x,y,\tau)=\dfrac{\partial u(x,y,\tau)}{\partial\tau}$.

Since finite difference method is used for (1), so let $k>0$ denoted time step and $h>0$ denoted space step, and $h=h_x=h_y$ where h_x represents step size in x-direction and h_y represents step size in y-direction. Define $x=ih,\ y=jh,$ $t_k=\tau k$ and $h=\dfrac{1}{n}$, where $\{i,j=0,1,2,\dots,n\},\ \ k=0,1,2,\dots,l$ and $n\in\mathbb{Z}^+$.

Consider a Taylor series expansion at point (x_i,y_j,t_k) for $u(x_i,y_j,t_k)$ is

$$u(x_i+h,y_j,t_k)=u_{i+1,j}^k=u_{i,j}^k+h\left.\frac{\partial u}{\partial x}\right|_{i,j}^k+\frac{h^2}{2}\left.\frac{\partial u}{\partial x}\right|_{i,j}^k+\frac{h^3}{6}\left.\frac{\partial u}{\partial x}\right|_{i,j}^k+\dots \quad (2)$$

$$u(x_i,y_j+h,t_k)=u_{i,j+1}^k=u_{i,j}^k+h\left.\frac{\partial u}{\partial y}\right|_{i,j}^k+\frac{h^2}{2}\left.\frac{\partial u}{\partial y}\right|_{i,j}^k+\frac{h^3}{6}\left.\frac{\partial u}{\partial y}\right|_{i,j}^k+\dots$$

(3)

If $\delta_x=$ central difference operator is introduced such as $\delta_x{}^2u_{i,j}^k=u_{i+1,j}^k-2u_{i,j}^k+u_{i-1,j}^k$ and $\delta_y{}^2u_{i,j}^k=u_{i,j+1}^k-2u_{i,j}^k+u_{i,j-1}^k$, then by using the Taylor series expansion at point $u_{i+1,j}^k$ and $u_{i,j+1}^k$ as defined above in (2) and (3)

$$\delta_x{}^2u_{i,j}^k=\left.\frac{\partial^2u}{\partial x^2}\right|_{i,j}^k+\frac{h^2}{12}\left.\frac{\partial^4u}{\partial x^4}\right|_{i,j}^k+\frac{h^4}{360}\left.\frac{\partial^6u}{\partial x^6}\right|_{i,j}^k+O(h^6) \quad (4)$$

$$\delta_y{}^2u_{i,j}^k=\left.\frac{\partial^2u}{\partial y^2}\right|_{i,j}^k+\frac{h^2}{12}\left.\frac{\partial^4u}{\partial y^4}\right|_{i,j}^k+\frac{h^4}{360}\left.\frac{\partial^6u}{\partial y^6}\right|_{i,j}^k+O(h^6) \quad (5)$$

After simplifying (4) and (5), we get

$$\left.\frac{\partial^2u}{\partial x^2}\right|_{i,j}^k=\left(1+\frac{1}{12}\delta_x{}^2\right)^{-1}\frac{\delta_x{}^2}{h^2}u_{i,j}^k+o(h^4) \quad (6)$$

$$\left.\frac{\partial^2u}{\partial y^2}\right|_{i,j}^k=\left(1+\frac{1}{12}\delta_y{}^2\right)^{-1}\frac{\delta_y{}^2}{h^2}u_{i,j}^k+o(h^4) \quad (7)$$

Caputo approximation formula is used for the time fraction derivative [32]

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x_i,y_j,t_{k+\frac{1}{2}})=a_1u_{i,j}^k+\sum_{s=1}^{k-1}(a_{k-s+1}-a_{k-s})u_{i,j}^s-a_su_{i,j}^0+\sigma\frac{(u_{i,j}^{k+1}+u_{i,j}^k)}{2^{1-\alpha}}+o(\tau^{2-\alpha})$$

$$\sigma=\frac{1}{\tau^\alpha\Gamma(2-\alpha)},a_s=\sigma((s+\frac{1}{2})^{1-\alpha}-(s-\frac{1}{2})^{1-\alpha}),\ s=0,1,2,\dots,n. \quad (8)$$

And average of function $u(x_i,y_j,t_k)$ at point $(i,j,k+\frac{1}{2})$ is

$$u_{i,j}^{k+\frac{1}{2}}=\frac{u_{i,j}^{k+1}+u_{i,j}^k}{2} \quad (9)$$

Since Crank Nicolson is the average of implicit and explicit schemes, so replacing k by $k+\frac{1}{2}$ in (6) and (7) and then substituting (6), (7), (8) and (9) in (1), we get

$$a_1u_{i,j}^k+\sum_{s=1}^{k-1}(a_{k-s+1}-a_{k-s})u_{i,j}^s-a_su_{i,j}^0+\sigma\frac{(u_{i,j}^{k+1}+u_{i,j}^k)}{2^{1-\alpha}}=$$
$$\left(1+\frac{1}{12}\delta_x{}^2\right)^{-1}\frac{\delta_x{}^2}{h^2}u_{i,j}^{k+\frac{1}{2}}+\left(1+\frac{1}{12}\delta_y{}^2\right)^{-1}\frac{\delta_y{}^2}{h^2}u_{i,j}^{k+\frac{1}{2}}$$
$$-\mu_0(\frac{u_{i,j}^{k+1}+u_{i,j}^k}{2})+f_{i,j}^{k+\frac{1}{2}}+o(\tau^{3-\alpha}+h^4)$$

$$a_1u_{i,j}^k+\sum_{s=1}^{k-1}(a_{k-s+1}-a_{k-s})u_{i,j}^s-a_su_{i,j}^0+\sigma\frac{u_{i,j}^{k+1}}{2^{1-\alpha}}+\sigma\frac{u_{i,j}^k}{2^{1-\alpha}}=$$
$$\frac{1}{h^2}\left(\left(1+\frac{1}{12}\delta_x{}^2\right)^{-1}\delta_x{}^2+\left(1+\frac{1}{12}\delta_y{}^2\right)^{-1}\delta_y{}^2\right)u_{i,j}^{k+\frac{1}{2}}$$
$$-\mu_0\frac{u_{i,j}^{k+1}}{2}-\mu_0\frac{u_{i,j}^k}{2}+f_{i,j}^{k+\frac{1}{2}}$$

and simplifying above

$$\sum_{s=1}^{k-1}(a_{k-s+1}-a_{k-s})u_{i,j}^s=\frac{1}{h^2}\left(\left(1+\frac{1}{12}\delta_x{}^2\right)^{-1}\delta_x{}^2+\left(1+\frac{1}{12}\delta_y{}^2\right)^{-1}\delta_y{}^2\right)u_{i,j}^{k+\frac{1}{2}}$$
$$-(\frac{1}{2}+\frac{\sigma}{2^{1-\alpha}})u_{i,j}^{k+1}-(\frac{1}{2}+(a_1-\frac{\sigma}{2^{1-\alpha}}))u_{i,j}^{k+1}+f_{i,j}^{k+\frac{1}{2}}$$

After rearranging and simplify for $u_{i,j}^{k+1}$, we get

$$
\begin{aligned}
&(2Gh^2+4A-4B)u_{i,j}^{k+1} = (A-2B)(u_{i+1,j}^{k+1}+u_{i-1,j}^{k+1}+u_{i,j+1}^{k+1}+u_{i,j-1}^{k+1}) \\
&+B(u_{i+1,j+1}^{k+1}+u_{i-1,j+1}^{k+1}+u_{i+1,j-1}^{k+1}+u_{i-1,j-1}^{k+1})+(4D-2Hh^2-4C)u_{i,j}^{k} \\
&+(C-2D)(u_{i+1,j}^{k}+u_{i-1,j}^{k}+u_{i,j+1}^{k}+u_{i,j-1}^{k})+D(u_{i+1,j+1}^{k}+u_{i-1,j+1}^{k}+ \\
&u_{i+1,j-1}^{k}+u_{i-1,j-1}^{k})+\frac{25}{18}h^2 f_{i,j}^{k+\frac{1}{2}}+\frac{5}{36}h^2(f_{i+1,j}^{k+\frac{1}{2}}+f_{i-1,j}^{k+\frac{1}{2}}+f_{i,j+1}^{k+\frac{1}{2}}+ \\
&f_{i,j-1}^{k+\frac{1}{2}})+\frac{h^2}{72}(f_{i+1,j+1}^{k+\frac{1}{2}}+f_{i-1,j+1}^{k+\frac{1}{2}}+f_{i+1,j-1}^{k+\frac{1}{2}}+f_{i-1,j-1}^{k+\frac{1}{2}}) \\
&+\sum_{s=1}^{k-1}(a_{k-s+1}-a_{k-s})(\frac{25}{18}h^2u_{i,j}^{s}+\frac{5}{36}h^2(u_{i+1,j}^{s}+u_{i-1,j}^{s}+u_{i,j+1}^{s} \\
&+u_{i,j-1}^{s})+\frac{h^2}{72}(u_{i+1,j+1}^{s}+u_{i-1,j+1}^{s}+u_{i+1,j-1}^{s}+u_{i-1,j-1}^{s}))
\end{aligned} \tag{10}
$$

where $G=\frac{1}{2}+\frac{\sigma}{2^{1-\alpha}}$, $H=\frac{1}{2}+(a_1-\frac{\sigma}{2^{1-\alpha}})$, $A=1-\frac{Gh^2}{6}$, $B=\frac{1}{6}-\frac{Gh^2}{72}$, $C=1-\frac{Hh^2}{6}$, $D=\frac{1}{6}-\frac{Hh^2}{72}$, $\sigma=\frac{1}{\tau^{\alpha}\Gamma(2-\alpha)}$ a

$a_s=\sigma((s+\frac{1}{2})^{1-\alpha}-(s-\frac{1}{2})^{1-\alpha})$.

Figure 1 represents the computational molecule of the high-order compact Crank Nicolson approximation equation (10). Figure 2 shows the nine point's high-order compact high order scheme for (1).

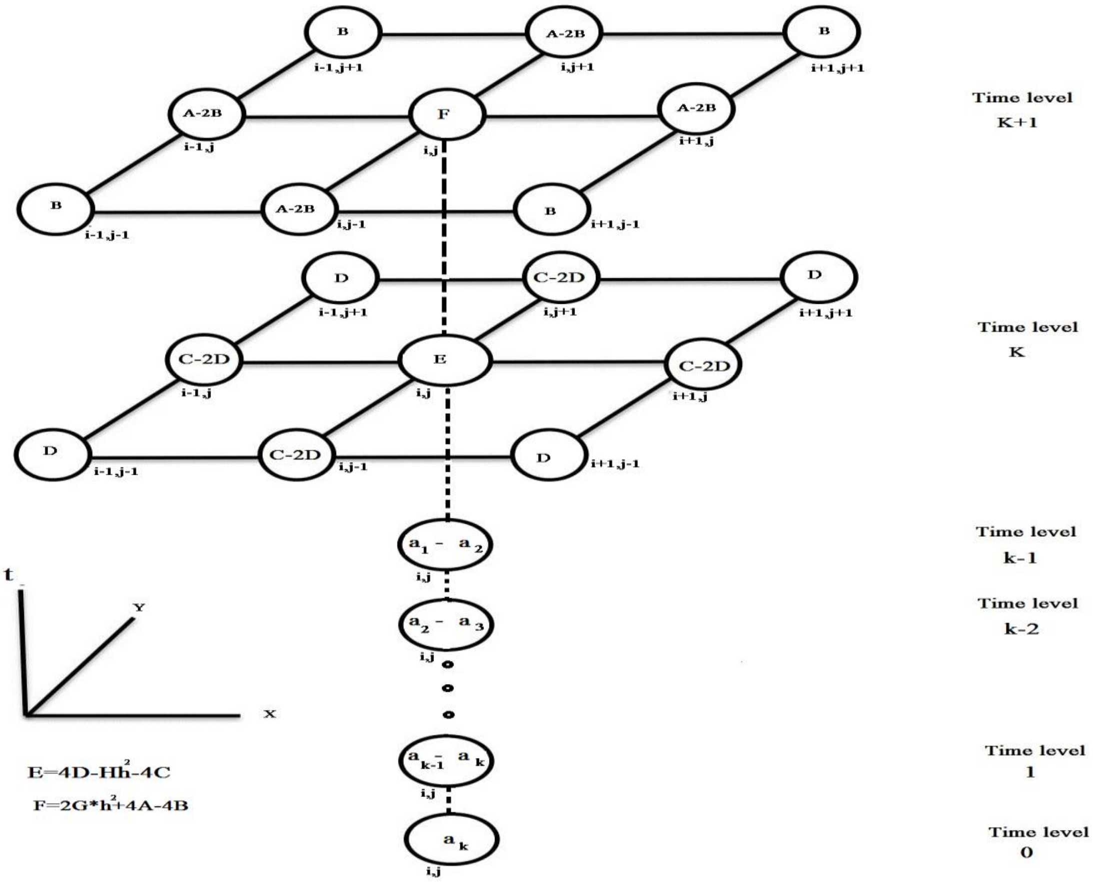

Figure 1. Computational molecule of high-order compact C-N scheme

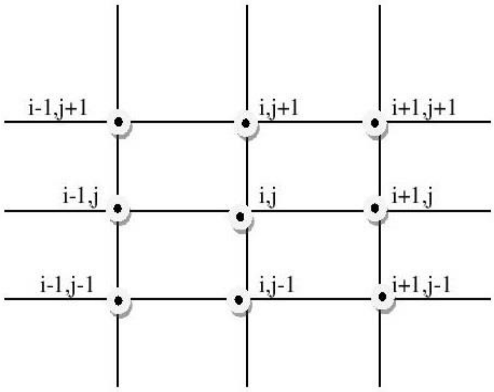

Figure 2. The Nine Grid Points involved in the Scheme

III. NUMERICAL EXAMPLES

To show the effectiveness of the proposed methods, we solved the two-dimensional time fractional cable equations with the help of PC with Core i7 Duo 3.40 GHz, 4GB of RAM with Window 7 operating system using Mathematica. The numerical examples were solved using a Successive over-relaxation (SOR) iterative method whilst using different time steps ($\tau =$ 0.25, 0.125, 0.083, 0.062, 0.05, 0.034, 0.016) for different mesh sizes ($n =$ 4, 8, 12, 16, 20, 30, 60) with $0 < t < 1$. Also for convergence criteria, tolerance $\varepsilon = 10^{-5}$ was used for the Maximum error (L_∞). We calculated the computational orders of convergence for the proposed method with the help of C_2-order of convergence [33]

$$C_2 - order = log_2\left(\frac{\| L_\infty(16\tau, 2h) \|}{\| L_\infty(\tau, h) \|}\right)$$

where τ represents time step, h represents space step and L_∞ represents maximum error, also $\|e\|_{l_\infty} = \max_{1\le i,j\le N-1}\left|U_{i,j}^k - u_{i,j}^k\right|$, where $U_{i,j}^k$ represents the exact solution while $u_{i,j}^k$ represents the approximate solution.

Example 1. Take the model problem [34]

$${}_0^C D_t^\alpha u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u + f(x,y,t),$$

where $f(x,y,t) = Sin(\pi x)Sin(\pi y)\left(t^2(1+2\pi^2) + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right)$ and having initial and boundary conditions

$$u(x,y,0) = 0,$$
$$u(0,y,t) = 0,\quad u(x,0,t) = 0,$$
$$u(1,y,t) = 0,\quad u(x,1,t) = 0,$$

with the exact solution $u(x,y,t) = t^2 Sin(\pi x)Sin(\pi y)$

Example 2. [35]

$${}_0^C D_t^\alpha u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u + f(x,y,t),$$

where $f(x,y,t) = Sin(\pi(\text{x}+\text{y}))\left(t^{1+\alpha}(2\pi^2+1) + \frac{\pi t\csc(\pi\alpha)}{\Gamma(-1-\alpha)}\right)$ and having initial and boundary conditions

$$u(x,y,0) = 0,$$
$$u(0,y,t) = e^y\, \text{t}^{1+3\alpha} + t^{1+\alpha}\sin(\pi\,\text{y}),\quad u(x,0,t) = e^x\, \text{t}^{1+3\alpha} + t^{1+\alpha}\sin(\pi x),$$
$$u(1,y,t) = e^{1+y}\, \text{t}^{1+3\alpha} + t^{1+\alpha}\sin(\pi(1+\text{y})),$$
$$u(x,1,\text{t}) = e^{1+x}\, \text{t}^{1+3\alpha} + t^{1+\alpha}\sin(\pi(x+1)),$$

with the exact solution $u(x,y,t) = t^{1+3\alpha}e^{x+y} + t^{1+\alpha}\sin(\pi(\text{x}+\text{y}))$

Example 3. [36]

$${}_0^C D_t^\alpha u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u + e^{x+y}\left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - t^2\right),$$

with initial and boundary conditions

$$u(x,y,0) = 0,$$
$$u(0,y,t) = t^2 e^y,\quad u(x,0,t) = t^2 e^x,$$
$$u(1,y,t) = t^2 e^{1+y},\quad u(x,1,t) = t^2 e^{x+1},$$

with exact solution $u(x,y,t) = t^2 e^{x+y}$.

Table 1. The number of iterations, Maximum error and Average error for example 1 when α = 0.5

τ	n	Iteration	Maximum error	Average error
0.25	4	48	6.2758 x 10^{-3}	40705 x 10^{-3}
0.125	8	43	3.3344 x 10^{-3}	17192 x 10^{-3}
0.083	12	38	1.7771 x 10^{-3}	8.4579 x 10^{-4}
0.062	16	42	1.0532 x 10^{-3}	4.8274 x 10^{-4}
0.05	20	45	6.9037 x 10^{-4}	3.0799 x 10^{-4}
0.034	30	40	3.1521 x 10^{-4}	1.3582 x 10^{-4}

Table 2. The number of iterations, Maximum error and Average error for example 2 when α = 0.5

τ	n	Iteration	Maximum error	Average error
0.25	4	51	9.7939 x 10^{-3}	5.0046 x10^{-3}
0.125	8	50	2.6916 x 10^{-3}	1.1738 x10^{-3}
0.083	12	54	1.0956 x 10^{-3}	5.2719 x10^{-4}
0.062	16	49	6.1573 x 10^{-4}	3.3637 x10^{-4}
0.05	20	55	4.6884 x 10^{-4}	2.4169 x10^{-4}
0.034	30	65	2.6788 x 10^{-4}	1.3297 x10^{-4}

Table 3. The number of iterations, Maximum error and Average error for example 3 when α = 0.5

τ	n	Iteration	Maximum error	Average error
0.25	4	51	9.9002 x 10^{-4}	6.2496 x 10^{-4}
0.125	8	51	5.0444 x 10^{-4}	2.1042 x 10^{-4}
0.083	12	55	4.3689 x 10^{-4}	2.0115 x 10^{-4}
0.062	16	50	3.2304 x 10^{-4}	1.5009 x 10^{-4}
0.05	20	55	2.3948 x 10^{-4}	1.1336 x 10^{-4}
0.034	30	67	1.3417 x 10^{-4}	6.5307 x 10^{-5}

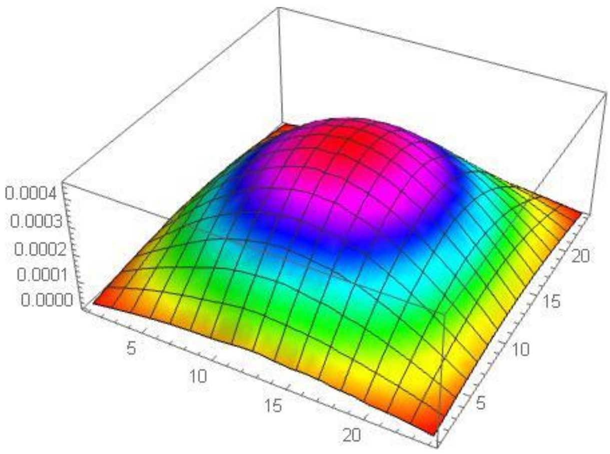

Figure 3. Absolute error $= |\text{exact-apprroximate}|$ for example 1

where $h = \tau = \frac{1}{25}$ and α =0.5

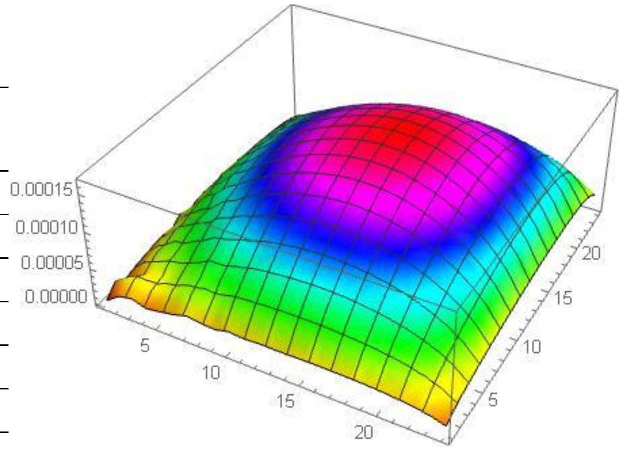

Figure 4. Absolute error $= |\text{exact-apprroximate}|$ for example 2

where $h = \tau = \frac{1}{25}$ and α =0.5

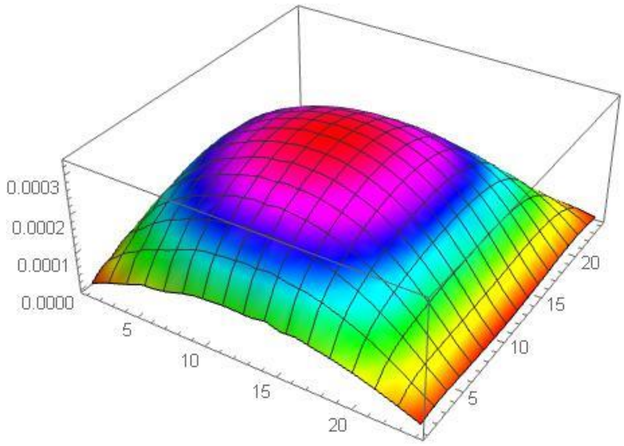

Figure 5. Absolute error $= |\text{exact-apprroximate}|$ for example 3

where $h = \tau = \frac{1}{25}$ and α =0.5

Table 4. C_2-order of convergence for example 2

$\alpha = 0.4$			$\alpha = 0.5$		
h/τ	Max error	C_2-order	h/τ	Max error	C_2-order
$h=\frac{1}{4}$, $\tau=\frac{1}{4}$	7.1737×10^{-2}	---	$h=\frac{1}{4}$, $\tau=\frac{1}{4}$	8.8157×10^{-2}	---
$h=\frac{1}{8}$, $\tau=\frac{1}{64}$	5.6927×10^{-3}	3.65	$h=\frac{1}{8}$, $\tau=\frac{1}{64}$	6.3467×10^{-3}	3.79
$\alpha = 0.6$			$\alpha = 0.7$		
h/τ	Max error	C_2-order	h/τ	Max error	C_2-order
$h=\frac{1}{4}$, $\tau=\frac{1}{4}$	1.0530×10^{-1}	---	$h=\frac{1}{4}$, $\tau=\frac{1}{4}$	1.2451×10^{-1}	---
$h=\frac{1}{8}$, $\tau=\frac{1}{64}$	7.2051×10^{-3}	3.86	$h=\frac{1}{8}$, $\tau=\frac{1}{64}$	8.3075×10^{-3}	3.91
$\alpha = 0.8$			$\alpha = 0.9$		
h/τ	Max error	C_2-order	h/τ	Max error	C_2-order
$h=\frac{1}{4}$, $\tau=\frac{1}{4}$	1.4784×10^{-1}	---	$h=\frac{1}{4}$, $\tau=\frac{1}{4}$	1.7843×10^{-1}	---
$h=\frac{1}{8}$, $\tau=\frac{1}{64}$	9.7198×10^{-3}	3.92	$h=\frac{1}{8}$, $\tau=\frac{1}{64}$	1.1875×10^{-2}	3.90

Table 5. C_2-order of convergence for example 3

$\alpha = 0.1$			$\alpha = 0.4$		
h/τ	Max error	C_2-order	h/τ	Max error	C_2-order
$h=\frac{1}{4}$, $\tau=\frac{1}{4}$	5.0916×10^{-4}	---	$h=\frac{1}{4}$, $\tau=\frac{1}{4}$	1.1470×10^{-3}	---
$h=\frac{1}{8}$, $\tau=\frac{1}{64}$	2.1087×10^{-5}	4.59	$h=\frac{1}{8}$, $\tau=\frac{1}{64}$	4.8156×10^{-5}	4.57
$h=\frac{1}{8}$, $\tau=\frac{1}{8}$	1.6115×10^{-4}	---	$h=\frac{1}{8}$, $\tau=\frac{1}{8}$	1.4206×10^{-4}	---
$h=\frac{1}{16}$, $\tau=\frac{1}{128}$	1.7018×10^{-5}	3.24	$h=\frac{1}{16}$, $\tau=\frac{1}{128}$	1.0974×10^{-5}	3.69
$\alpha = 0.5$			$\alpha = 0.6$		
h/τ	Max error	C_2-order	h/τ	Max error	C_2-order
$h=\frac{1}{4}$, $\tau=\frac{1}{4}$	9.9002×10^{-4}	---	$h=\frac{1}{4}$, $\tau=\frac{1}{4}$	9.9013×10^{-4}	---
$h=\frac{1}{8}$, $\tau=\frac{1}{64}$	5.7605×10^{-5}	4.10	$h=\frac{1}{8}$, $\tau=\frac{1}{64}$	4.7605×10^{-5}	4.37
$h=\frac{1}{8}$, $\tau=\frac{1}{8}$	5.0444×10^{-4}	---	$h=\frac{1}{8}$, $\tau=\frac{1}{8}$	8.6195×10^{-4}	---
$h=\frac{1}{16}$, $\tau=\frac{1}{128}$	2.8096×10^{-5}	4.16	$h=\frac{1}{16}$, $\tau=\frac{1}{128}$	3.8712×10^{-5}	4.47

From table 1-3 compact nine point's scheme shows that with increasing mesh size maximum and average error are reduced, which shows that reliability and accuracy of the proposed scheme. Figure 3-5 shows the 3-D graphs of Absolute error for example 1 and example 2 for the different values of $h, \tau,$ and $\alpha,$ which shows the accuracy of the proposed scheme. In Table 4 and Table 5 C_2- order of convergence is checked for the different values of $\alpha's$ for example 2 and example 3 respectively; it is observed that the experimental spatial convergence order of the proposed scheme is approximately four.

IV. CONCLUSION

We have presented a new high-order compact Crank Nicolson scheme for the solution of two- dimensional time fractional cable equations. It is observed that the proposed scheme is accurate and reliable. The proposed scheme has the theoretical order of convergence $O(\tau^{2-\alpha}+h^4)$. C_2-order of convergence for the different values of $\alpha's$ shows that the spatial accuracy of the scheme agrees with the theoretical spatial order of convergence.

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