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A Single Convergent Control Parameter Optimal Homotopy Asymptotic Method Approximate-Analytical Solution of Fuzzy Heat Equation

Sarmad Altaie[1,2*], Ali Fareed Jameel[1] and Azizan Saaban[1]

[1]*School of Quantitative Sciences, Universiti Utara Malaysia (UUM), Kedah, Sintok, 06010 Malaysia*

[2]*Computer Engineering Department, University of Technology-Iraq*

To solve fuzzy partial differential equations (FPDE), we develop an approximate analytical method which is based on Optimal Homotopy Asymptotic Method (OHAM). The method with a Single convergent control parameter has been applied to Fuzzy Heat Equation (FHE) with fuzzy initial condition. An illustrative example has been given to demonstrate the accuracy, efficiency, and flexibility of the proposed method.

Keywords: optimal homotopy asymptotic method; single convergent control parameter; fuzzy partial differential equations; fuzzy heat equation

I. INTRODUCTION

Fuzzy differential equations (FDEs) form a major portion of a fuzzy analysis theory and are a useful tool to describe a dynamic phenomenon if its nature and data is vague (Ali *et. al.*, 2018). The recurrent participating in the modeling of numerous industrialized applications, such as electromagnetic fields, dynamics of structures, biomechanics, heat transfer, and many others, brought fuzzy partial differential equations (FPDEs) to be under consideration by the scientific, and engineering communities. Yielding a demand on finding methods to solve these equations since the exact solutions are rarely available. The numerical and approximate-analytical methods for FPDEs have been attempted by numerous authors like (Allahviranloo, 2002; Olver, 2014; Jameel *et. al.*, 2016; Nemati & Matinfar, 2008; Corveleyn *et. al.*, 2010; Farajzadeh *et. al.*, 2010; Mikaeilvand & Khakrangin, 2012; Behzadi, 2013; Pirzada, & Vakaskar, 2015) yet the field still lacking for further accurate and capable solutions.

In this work, we present a method based on OHAM with a Single Control Parameter to obtain approximate-analytical solution for FHE with fuzzy initial condition. The outline of this paper is as follows: Section II will show the development of OHAM for solving FPDEs. Further details of the fuzzy heat equation will be presented in Section III. In Section IV the capabilities of the developed OHAM is illustrated through application with a Single Control Parameter for solving fuzzy heat equation, and finally in Section V, and VI the results and conclusion of the work.

II. OHAM MATHEMATICAL FORMULATION

The OHAM has been applied to derive an approximate-analytical solution of linear and nonlinear time dependent PDEs in (Hussian & Suhhiem, 2016; Iqbal *et. al.* 2010). In this section, it will be applied to the subsequent FPDE,

$$\mathcal{L}(\tilde{v}(s,t;\alpha)) + \mathcal{N}(\tilde{v}(s,t;\alpha)) + \tilde{\mathcal{F}}(s,t;\alpha) = 0 \qquad s \in \Omega \quad (1)$$

$$\mathcal{B}\left(\tilde{v}(s,t;\alpha), \frac{\partial \tilde{v}(s,t;\alpha)}{\partial t}\right) = 0 \qquad s \in \Gamma$$

where $\tilde{v}(s,t;\alpha)$ is an unknown fuzzy function with independent variables s and time variable t, $\mathcal{L}$ is a linear operator, $\mathcal{N}$ is a nonlinear operator, $\tilde{\mathcal{F}}(s,t;\alpha)$ is a known fuzzy source of nonhomogeneity, the boundary operator is $\mathcal{B}$, and Γ is the boundary of the domain Ω. First, construct a family of equations,

$$(1-p)\left[\mathcal{L}\left(\tilde{\Phi}(s,t;\alpha;p)\right) + \tilde{\mathcal{F}}(s,t;\alpha)\right] \quad (2)$$

$$= \tilde{\mathcal{H}}(\alpha;p)\left[\mathcal{L}\left(\tilde{\Phi}(s,t;\alpha;p)\right) + \mathcal{N}\left(\tilde{\Phi}(s,t;\alpha;p)\right) + \tilde{\mathcal{F}}(s,t;\alpha)\right]$$

$$\mathcal{B}\left(\tilde{\Phi}(s,t;\alpha;p), \frac{\partial \tilde{\Phi}(s,t;\alpha;p)}{\partial t}\right) = 0$$

where $p \in [0,1]$ is an embedding parameter, $\tilde{\mathcal{H}}(\alpha;p)$ is a nonzero auxiliary fuzzy function for $p \neq 0$, and $\tilde{\mathcal{H}}(\alpha;0) = 0$, $\tilde{\Phi}(s,t;\alpha;p)$ is an unknown fuzzy function. Apparently, once

*Corresponding author's e-mail: 120024@uotechnology.edu.iq

$p = 0$ and $p = 1$,

$$\tilde{\Phi}(s,t;\alpha;0) = \tilde{v}_0(s,t;\alpha) \quad (3)$$

$$\tilde{\Phi}(s,t;\alpha;1) = \tilde{v}(s,t;\alpha). \quad (4)$$

Consequently, as p escalates in the interval $(0,1)$, $\tilde{\Phi}(s,t;\alpha;p)$ changes from $v_0(s,t)$ to $v(s,t)$. Furthermore, the zeroth order problem $\tilde{v}_0(s,t;\alpha)$ is acquired from equation (2) for $p = 0$,

$$\mathcal{L}(\tilde{v}_0(s,t;\alpha)) + \tilde{\mathcal{F}}(s,t;\alpha) = 0 \quad (5)$$

$$\mathcal{B}\left(\tilde{v}_0(s,t;\alpha), \frac{\partial \tilde{v}_0(s,t;\alpha)}{\partial t}\right) = 0$$

Now, the auxiliary fuzzy function $\tilde{\mathcal{H}}(\alpha;p)$ is selected from the formula,

$$\tilde{\mathcal{H}}(\alpha;p) = \sum_{i=1}^{\infty} \tilde{C}_i(\alpha)\, p^i = \left[\sum_{i=1}^{\infty} \underline{C}_i(\alpha)\, p^i, \sum_{i=1}^{\infty} \overline{C}_i(\alpha)\, p^i\right] \quad (6)$$

the fuzzy constants $\tilde{C}_1, \tilde{C}_2, \ldots$ should be identified for each α-level. Consider the solution of eq. (2),

$$\tilde{\Phi}(s,t,\tilde{C}_i;\alpha;p) = \tilde{v}_0(s,t;\alpha) + \sum_{k\geq 1} \tilde{v}_k(s,t,\tilde{C}_i;\alpha)p^k \quad (7)$$

where $= 1,2,\cdots$.

Comparing the coefficients of identical powers of p after substituting equation (7) into equation (2), gives the governing equation of $\tilde{v}_j(s,t;\alpha)$,

$$\mathcal{L}(\tilde{v}_1(s,t;\alpha)) = \tilde{C}_1(\alpha)\mathcal{N}_0(\tilde{v}_0(s,t;\alpha)) \quad (8)$$

$$\mathcal{B}\left(\tilde{v}_1(s,t;\alpha), \frac{\partial \tilde{v}_1(s,t;\alpha)}{\partial t}\right) = 0$$

$$\mathcal{L}(\tilde{v}_j(s,t;\alpha) - \tilde{v}_{j-1}(s,t;\alpha)) = \tilde{C}_j(\alpha)\mathcal{N}_0(\tilde{v}_0(s,t;\alpha)) \quad (9)$$

$$+ \sum_{i=1}^{j-1} \tilde{C}_i(\alpha)\left[\mathcal{L}\left(\tilde{v}_{j-i}(s,t;\alpha)\right) + \mathcal{N}_{j-i}\left(\tilde{v}_0(s,t;\alpha), \tilde{v}_1(s,t;\alpha), \cdots, \tilde{v}_{j-1}(s,t;\alpha)\right)\right]$$

$$\mathcal{B}\left(\tilde{v}_j(s,t;\alpha), \frac{\partial \tilde{v}_j(s,t;\alpha)}{\partial t}\right) = 0 \quad j = 2,3,\cdots$$

where $\mathcal{N}_m(\tilde{v}_0(s,t;\alpha), \tilde{v}_1(s,t;\alpha), \cdots, \tilde{v}_m(s,t;\alpha))$ is the coefficient of p^m, acquired by expanding in series $\mathcal{N}[\phi(s,t,\tilde{C}_i;\alpha;p)]$ with respect to p,

$$\mathcal{N}[\phi(s,t,\tilde{C}_i;\alpha;p)] = \mathcal{N}_0(\tilde{v}_0(s,t;\alpha)) \quad (10)$$

$$+ \sum_{m\geq 1} \mathcal{N}_m(\tilde{v}_0(s,t;\alpha), \tilde{v}_1(s,t;\alpha), \cdots, \tilde{v}_m(s,t;\alpha))p^m$$

where $\tilde{\Phi}(s,t,\tilde{C}_i;\alpha;p)$ is specified by equation (7).

An emphasis has to be given to the fact that $v_k(x)$ for $k \geq 0$ are governed by the linear equations (5), (8) and (9) with the original problem's linear boundary conditions. The convergence of the series (7) depends upon the auxiliary constants $\tilde{C}_1, \tilde{C}_2, \ldots$. If it is convergent at $p = 1$, then

$$\tilde{v}(s,t,\tilde{C}_i(\alpha);\alpha) = \tilde{v}_0(s,t;\alpha) + \sum_{k\geq 1} v_k(s,t,\tilde{C}_i(\alpha);\alpha) \quad (11)$$

The approximate solution of equation (1) can be specified in the form,

$$\overset{(m)}{\tilde{v}}(s,t,\tilde{C}_i(\alpha);\alpha) = \tilde{v}_0(s,t;\alpha) + \sum_{k=1}^{m} \tilde{v}_k(s,t,\tilde{C}_i(\alpha);\alpha) \quad (12)$$

where $i = 1,2,\cdots m$.

Substituting equation (12) into equation (1), results in the following expression of the residual,

$$\tilde{\rho}(s,t,\tilde{C}_i(\alpha);\alpha) = \quad (13)$$

$$\mathcal{L}\left(\overset{(m)}{\tilde{v}}(s,t,\tilde{C}_i(\alpha);\alpha)\right) + \mathcal{N}\left(\overset{(m)}{\tilde{v}}(s,t,\tilde{C}_i(\alpha);\alpha)\right) + \tilde{\mathcal{F}}(s,t;\alpha)$$

where $i = 1,2,\cdots m$.

If $\tilde{\rho}(s,t,\tilde{C}_i(\alpha);\alpha) = 0$ then $\overset{(m)}{\tilde{v}}(s,t,\tilde{C}_i(\alpha);\alpha)$ is the exact solution, and this does not occur for nonlinear problems. According to Idrees *et. al.*, 2012, the least squares method can be employed here to minimize the functional,

$$\delta\left(\tilde{C}_i(\alpha)\right) = \iint_a^b \tilde{\rho}^2(s,t,\tilde{C}_i(\alpha);\alpha)\, ds\, dt \quad (14)$$

where a and b are values dependent on the boundaries of the problem. The unknown constants $\tilde{C}_i(\alpha)$ $(i = 1,2,\cdots m)$ can be identified optimally for error minimization from the conditions,

$$\frac{\partial \delta}{\partial \tilde{C}_1(\alpha)} = \frac{\partial \delta}{\partial \tilde{C}_2(\alpha)} = \cdots = \frac{\partial \delta}{\partial \tilde{C}_m(\alpha)} = 0 \quad (15)$$

the approximate solution (of order m) (12) is well-determined with these constants known.

III. FUZZY HEAT EQUATION

Consider the model and information available by Allahviranloo, 2002 and Altaie *et. al.*, 2017, a general model for the fuzzy heat equation will be written and analyzed using the properties of the fuzzy set theory. Consider $0 < x < l$, $0 < t < T$,

$$\frac{\partial}{\partial t}\tilde{v}(x,t) = \tilde{\mathrm{H}}(x)\frac{\partial^2}{\partial x^2}\tilde{v}(x,t) + \tilde{\Lambda}(x,t) \quad (16)$$

$$\tilde{v}(x,0) = \tilde{\varphi}(x) \qquad 0 \leq x \leq l$$

In this model, $\tilde{v}(x,t)$ is a fuzzy function with crisp variables x and t. Furthermore, $\frac{\partial}{\partial t}\tilde{v}(x,t)$, $\frac{\partial^2}{\partial x^2}\tilde{v}(x,t)$ are fuzzy partial derivatives in the Hukuhara sense. In addition, $\tilde{H}(x) = \tilde{\gamma}_1\mathrm{H}(x)$ is a fuzzy function of crisp variables represent the thermal diffusivity, $\tilde{\Lambda}(x,t) = \tilde{\gamma}_2\Lambda(x,t)$ is a fuzzy function of crisp variables as a nonhomogeneous term. Moreover, $\tilde{v}(x,0)$ is the fuzzy initial condition with fuzzy function of crisp variables $\tilde{\varphi}(x) = \tilde{\gamma}_3\varphi(x)$. Finally, $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ are convex fuzzy numbers, and $\mathrm{H}(x)$, $\Lambda(x,t)$, $\varphi(x)$ are crisp functions. The defuzzification of this model for all $\alpha \in [0,1]$ as follows,

$$[\tilde{v}(x,t)]_\alpha = [\underline{v}(x,t;\alpha), \overline{v}(x,t;\alpha)],$$

$\left[\frac{\partial}{\partial t}\tilde{v}(x,t)\right]_\alpha = \left[\frac{\partial}{\partial t}\underline{v}(x,t;\alpha), \frac{\partial}{\partial t}\overline{v}(x,t;\alpha)\right]$,

$\left[\frac{\partial^2}{\partial x^2}\tilde{v}(x,t)\right]_\alpha = \left[\frac{\partial^2}{\partial x^2}\underline{v}(x,t;\alpha), \frac{\partial^2}{\partial x^2}\overline{v}(x,t;\alpha)\right]$,

$[\tilde{H}(x)]_\alpha = [\underline{H}(x;\alpha), \overline{H}(x;\alpha)]$, $\tilde{\gamma}_1 = [\underline{\gamma}_1(\alpha), \overline{\gamma}_1(\alpha)]$,

$[\tilde{\Lambda}(x,t)]_\alpha = [\underline{\Lambda}(x,t;\alpha), \overline{\Lambda}(x,t;\alpha)]$, $\tilde{\gamma}_2 = [\underline{\gamma}_2(\alpha), \overline{\gamma}_2(\alpha)]$,

$[\tilde{u}(x,0)]_\alpha = [\underline{u}(x,0;\alpha), \overline{u}(x,0;\alpha)]$, $[\tilde{\varphi}(x)]_\alpha = [\underline{\varphi}(x;\alpha), \overline{\varphi}(x;\alpha)]$,

$\tilde{\gamma}_3 = [\underline{\gamma}_3(\alpha), \overline{\gamma}_3(\alpha)]$

Now, using the extension principle, the membership function defined as follows,

$\underline{v}(x,t;\alpha) = min\{\tilde{v}(t,\tilde{\mu}(\alpha)) | \tilde{\mu}(\alpha) \in \tilde{v}(x,t;\alpha)\}$

$\overline{v}(x,t;\alpha) = max\{\tilde{v}(t,\tilde{\mu}(\alpha)) | \tilde{\mu}(\alpha) \in \tilde{v}(x,t;\alpha)\}$

Hence, we can rewrite equation (16) for $0 < x < l$, $0 < t < T$ and $\alpha \in [0,1]$ as,

$$\frac{\partial}{\partial t}\underline{v}(x,t;\alpha) - \underline{\gamma}_1(\alpha)H(x)\frac{\partial^2}{\partial x^2}\underline{v}(x,t;\alpha) - \underline{\gamma}_2(\alpha)\Lambda(x,t) = 0 \quad (17a)$$

$\underline{v}(x,0;\alpha) = \underline{\gamma}_3(\alpha)\varphi(x)$

$$\frac{\partial}{\partial t}\overline{v}(x,t;\alpha) - \overline{\gamma}_1(\alpha)H(x)\frac{\partial^2}{\partial x^2}\overline{v}(x,t;\alpha) - \overline{\gamma}_2(\alpha)\Lambda(x,t) = 0 \quad (17b)$$

$\overline{v}(x,0;\alpha) = \overline{\gamma}_3(\alpha)\varphi(x)$

IV. ILLUSTRATIVE EXAMPLE

Applying OHAM with a single control parameter have been used by Jameel *et. al.*, 2016, for solving Fuzzy Ordinary Differential Equations. Here a tenth order solution with single control parameter have been applied to solve Fuzzy Heat equation with fuzzy initial condition. Consider the following fuzzy heat equation with the given initial condition, and $0 < x < 1$, $0 < t < 1$,

$$\frac{\partial}{\partial t}\tilde{v}(x,t) = \frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\tilde{v}(x,t) \quad (18)$$

$\tilde{u}(x,0) = \tilde{\varphi}(x) = \tilde{\gamma}_3(\alpha)x^2 \quad 0 \le x \le 1$

where $\tilde{\gamma}_3(\alpha) = [\underline{\gamma}(\alpha), \overline{\gamma}(\alpha)] = [\alpha - 1, 1 - \alpha]$.

Now, the defuzzification of this model for all $\alpha \in [0,1]$,

$[\tilde{v}(x,t)]_\alpha = [\underline{v}(x,t;\alpha), \overline{v}(x,t;\alpha)]$,

$\left[\frac{\partial}{\partial t}\tilde{v}(x,t)\right]_\alpha = \left[\frac{\partial}{\partial t}\underline{v}(x,t;\alpha), \frac{\partial}{\partial t}\overline{v}(x,t;\alpha)\right]$,

$\left[\frac{\partial^2}{\partial x^2}\tilde{v}(x,t)\right]_\alpha = \left[\frac{\partial^2}{\partial x^2}\underline{v}(x,t;\alpha), \frac{\partial^2}{\partial x^2}\overline{v}(x,t;\alpha)\right]$,

$[\tilde{H}(x)]_\alpha = \frac{1}{2}x^2$,

$[\tilde{v}(x,0)]_\alpha = [\underline{v}(x,0;\alpha), \overline{v}(x,0;\alpha)] = [(\alpha-1)x^2, (1-\alpha)x^2]$

Hence, we can rewrite equation (18) for $0 < x < 1$, $0 < t < 1$ and $\alpha \in [0,1]$ as,

$$\frac{\partial}{\partial t}\underline{v}(x,t;\alpha) - \frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\underline{v}(x,t;\alpha) = 0 \quad (19a)$$

$\underline{v}(x,0;\alpha) = (\alpha-1)x^2$

$$\frac{\partial}{\partial t}\overline{v}(x,t;\alpha) - \frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\overline{v}(x,t;\alpha) = 0 \quad (19b)$$

$\overline{v}(x,0;\alpha) = (1-\alpha)x^2$

Applying the method developed in section II with OHAM solution of the 10th order and single convergent control parameters, and using equation (19a) and (19b), leads to the following for the lower problem,

$$\mathcal{L}(\underline{v}(x,t;\alpha)) = \frac{\partial}{\partial t}\underline{v}(x,t;\alpha), \quad (20a)$$

$\mathcal{N}(\underline{v}(x,t;\alpha)) = -\frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\underline{v}(x,t;\alpha)$,

$\underline{v}(x,0;\alpha) = (\alpha-1)x^2$

also, the following for the upper problem,

$$\mathcal{L}(\overline{v}(x,t;\alpha)) = \frac{\partial}{\partial t}\overline{v}(x,t;\alpha), \quad (20b)$$

$\mathcal{N}(\overline{v}(x,t;\alpha)) = -\frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\overline{v}(x,t;\alpha)$,

$\overline{v}(x,0;\alpha) = (1-\alpha)x^2$

The 0th order problem is,

$$\frac{\partial}{\partial t}\underline{v}_0(x,t;\alpha) = 0 \quad (21a)$$

$\underline{v}_0(x,0;\alpha) = (\alpha-1)x^2$

$$\frac{\partial}{\partial t}\overline{v}_0(x,t;\alpha) = 0 \quad (21b)$$

$\overline{v}_0(x,0;\alpha) = (1-\alpha)x^2$

The 1st order problem,

$$\frac{\partial}{\partial t}\underline{v}_1(x,t,\underline{C}_1(\alpha);\alpha) - \frac{\partial}{\partial t}v_0(x,t;\alpha) \quad (22a)$$

$= \underline{C}_1(\alpha)\left[\frac{\partial}{\partial t}\underline{v}_0(x,t;\alpha) - \frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\underline{v}_0(x,t;\alpha)\right]$

$\underline{v}_1(x,0;\alpha) = 0$

$$\frac{\partial}{\partial t}\overline{v}_1(x,t,\overline{C}_1(\alpha);\alpha) - \frac{\partial}{\partial t}\overline{v}_0(x,t;\alpha) \quad (22b)$$

$= \overline{C}_1(\alpha)\left[\frac{\partial}{\partial t}\overline{v}_0(x,t;\alpha) - \frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\overline{v}_0(x,t;\alpha)\right]$

$\overline{v}_1(x,0;\alpha) = 0$

The nth problem is,

$\frac{\partial}{\partial t}\underline{v}_n(x,t,\underline{C}_1(\alpha);\alpha) - \frac{\partial}{\partial t}\underline{v}_{n-1}(x,t,\underline{C}_1(\alpha);\alpha)$

$= \underline{C}_1(\alpha)\left[\frac{\partial}{\partial t}\underline{v}_{n-1}(x,t,\underline{C}_1(\alpha);\alpha) - \right.$

$$\left.\frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\underline{v}_{n-1}(x,t,\underline{C}_1(\alpha);\alpha)\right] \quad (23a)$$

$\underline{v}_n(x,0;\alpha) = 0$

$\frac{\partial}{\partial t}\overline{v}_n(x,t,\overline{C}_1(\alpha);\alpha) - \frac{\partial}{\partial t}\overline{v}_{n-1}(x,t,\overline{C}_1(\alpha);\alpha)$

$= \overline{C}_1(\alpha)\left[\frac{\partial}{\partial t}\overline{v}_{n-1}(x,t,\overline{C}_1(\alpha);\alpha) - \right.$

$$\left.\frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\overline{v}_{n-1}(x,t,\overline{C}_1(\alpha);\alpha)\right] \quad (23b)$$

$\overline{v}_n(x,0;\alpha) = 0$

where $n = 2,3,\ldots,10$.

The solution of equation (19a) and (19b) can be determined approximately as,

$$\underline{v}^{(10)}(x,t,\underline{C}_1(\alpha);\alpha) \quad (24a)$$

$$= \underline{v}_0(x,t;\alpha) + \underline{v}_1(x,t,\underline{C}_1(\alpha);\alpha) + \underline{v}_2(x,t,\underline{C}_1(\alpha);\alpha) + \underline{v}_3(x,t,\underline{C}_1(\alpha);\alpha) + \underline{v}_4(x,t,\underline{C}_1(\alpha);\alpha) + \underline{v}_5(x,t,\underline{C}_1(\alpha);\alpha) + \underline{v}_6(x,t,\underline{C}_1(\alpha);\alpha) + \underline{v}_7(x,t,\underline{C}_1(\alpha);\alpha) + \underline{v}_8(x,t,\underline{C}_1(\alpha);\alpha) + \underline{v}_9(x,t,\underline{C}_1(\alpha);\alpha) + \underline{v}_{10}(x,t,\underline{C}_1(\alpha);\alpha)$$

$$\overline{v}^{(10)}(x,t,\overline{C}_1(\alpha);\alpha) \quad (24b)$$

$$= \overline{v}_0(x,t;\alpha) + \overline{v}_1(x,t,\overline{C}_1(\alpha);\alpha) + \overline{v}_2(x,t,\overline{C}_1(\alpha);\alpha) + \overline{v}_3(x,t,\overline{C}_1(\alpha);\alpha) + \overline{v}_4(x,t,\overline{C}_1(\alpha);\alpha) + \overline{v}_5(x,t,\overline{C}_1(\alpha);\alpha) + \overline{v}_6(x,t,\overline{C}_1(\alpha);\alpha) + \overline{v}_7(x,t,\overline{C}_1(\alpha);\alpha) + \overline{v}_8(x,t,\overline{C}_1(\alpha);\alpha) + \overline{v}_9(x,t,\overline{C}_1(\alpha);\alpha) + \overline{v}_{10}(x,t,\overline{C}_1(\alpha);\alpha)$$

Now, we use the Least Squares Method to evaluate the constants $\underline{C}_1(\alpha)$, and $\overline{C}_1(\alpha)$ for each selected $\alpha \in [0,1]$.

V. RESULT

The results listed in table 1 & 2, shows a 10th order OHAM solution, the single convergence control parameter value, and accuracy.

Table 1. lower solution and accuracy of 10th order OHAM with single convergence control parameter at t = 0.6, x = 0.4 for all α ∈ [0, 1)

α	$\underline{C}_1$	$\underline{v}_{OHAM}$	E_{OHAM}
0	-1.033685	-0.29153901	1.1157741×10^{-14}
0.2	-0.98034974	-0.23323121	1.7693896×10^{-10}
0.4	-1.4325154	-0.17492163	1.7751545×10^{-6}
0.6	-0.98034974	-0.1166156	8.8469482×10^{-11}
0.8	-0.93574959	-0.0583078	1.6842284×10^{-9}
0.9	-1.4259717	-0.00029154	2.4648947×10^{-9}

Table 2. upper solution and accuracy of 10th order OHAM with single convergence control parameter at $t = 0.6$, $x = 0.4$ for all $\alpha \in [0, 1)$

α	$\overline{C}_1$	$\overline{v}_{OHAM}$	E_{OHAM}
0	-1.033685	0.29153901	1.1157741×10^{-14}
0.2	-0.98034974	0.23323121	1.7693896×10^{-10}
0.4	-1.4325154	0.17492163	1.7751545×10^{-6}
0.6	-0.98034974	0.1166156	8.8469482×10^{-11}
0.8	-0.93574959	0.0583078	1.6842284×10^{-9}
0.9	-1.4259717	0.00029154	2.4648947×10^{-9}

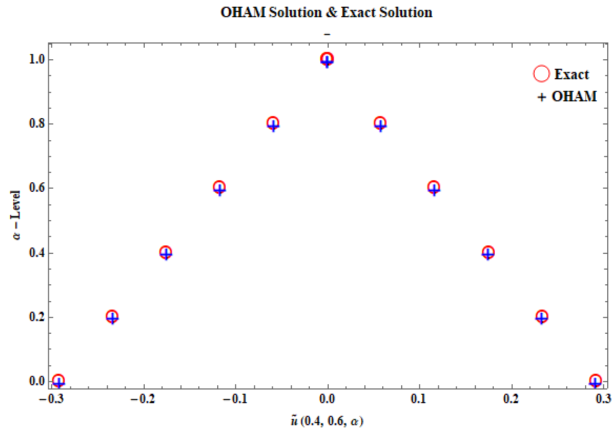
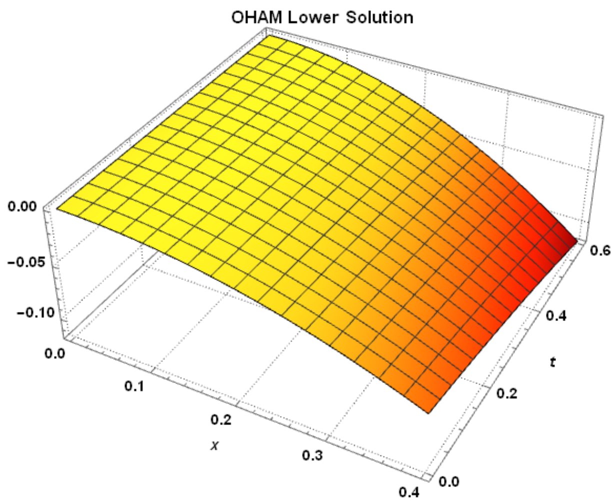


Figure 1. The 10th order OHAM solution with single convergence control parameter and exact solution at at $x = 0.4$, $t = 0.6$, for all $\alpha \in [0, 1)$



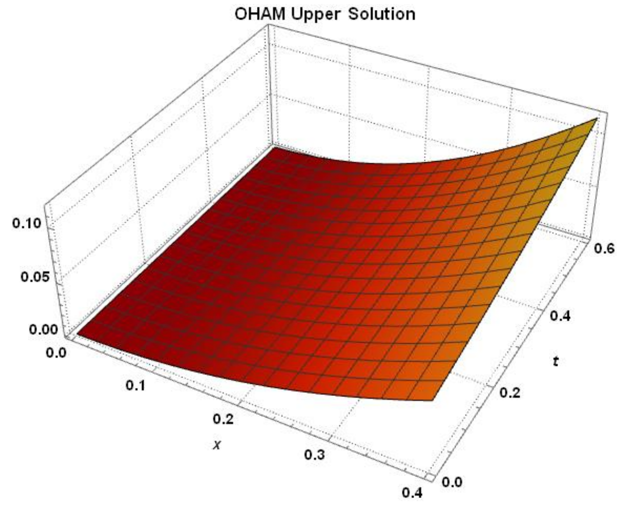


Figure 2. The OHAM lower and upper solution at $\alpha = 0.6$

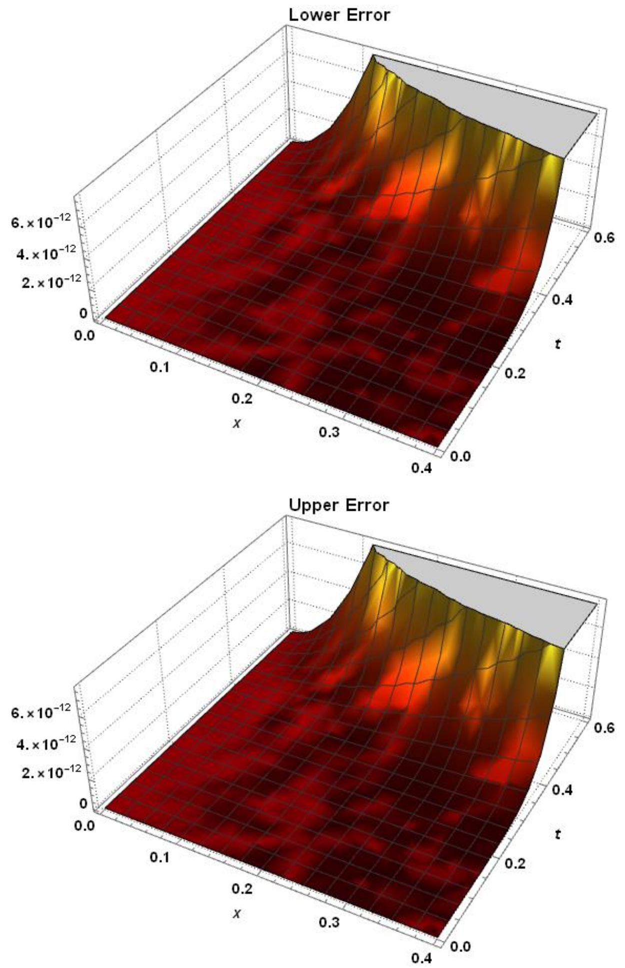


Figure 3. The OHAM lower and upper solution error at $\alpha = 0.6$

VI. CONCLUSION

This research has met its key objective by developing OHAM to derive an approximate-analytical solution of FPDEs, and then applying it with single convergent control parameter to obtain the solution to FHE, figure 2. The results as shown in table 1 & 2, in addition to figures 1 & 3, exhibited a high accuracy, less complexity, and computational time, which is very helpful for solving FPDEs in scientific and engineering applications.
The solution as shown by figure 1, acquired have the shape of triangular fuzzy numbers and hence the developed method fulfil the fuzzy numbers properties.

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