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A Stable Fourth Order Block Backward Differentiation Formulas (α) for Solving Stiff Initial Value Problems

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In this paper, an order four block backward differentiation formulas with independent parameter α, BBDF(α) are developed for the numerical solution of stiff ordinary differential equations (ODEs). The order and stability analysis are discussed and the BBDF (α) is shown to be A-stable, which is the requirement for solving stiff ODEs. Numerical results show the advantage of the BBDF (α) as compared to the existing methods in terms of its accuracy.

Keywords: block; stiff; ordinary differential equations

I. INTRODUCTION

Differential equations have long been an essential part in most branches of physical sciences and engineering all over the world. One of the famous differential equations is ordinary differential equations (ODEs). The general form of first order ODEs is defined as

$$y' = f(x,y), \quad y(a) = y_0, \tag{1}$$

where the interval is $x \in [a,b]$. The systems of (1) are said to be stiff if the eigenvalues of the matrix $\frac{\partial f}{\partial y}$ have negative real parts at every time *x* and varies greatly in magnitude (Lambert, 1991).

The backward differentiation formula (BDF) plays a special role in the numerical solutions of stiff ODEs. This method was introduced by Gear (1971) and has been expanded gradually by Byrne and Hindmarsh (1975) and Shampine and Reichelt (1997) using several approaches. The BDF is known as non-block method because it computes only one approximated solution for each step. In line with this, many researchers such as Ibrahim *et al.* (2007), Nasir *et al.* (2012), Abasi *et al.* (2014) and Zawawi *et al.* (2015) have developed various classes of block backward differentiation formulas (BBDF) to produce multiple approximated solutions simultaneously.

The aim of this paper is to present the numerical solutions of stiff ODEs using an order four block backward differentiation formula with independent parameter α, namely BBDF(α). The formulation of the method, order, stability analysis, implementation and numerical experiments will be discussed in the following sections.

II. FORMULATION OF THE METHOD

The method is formulated using constant step size, h where earlier block consists of three previous points, x_n, x_{n-1} and x_{n-2} to compute two solutions, y_{n+1} and y_{n+2} at two points, x_{n+1} and x_{n+2} concurrently. The interpolation polynomial is determined using Lagrange polynomial, $P_k(x)$ of degree k which is defined as follows:

$$P_k(x) = \sum_{j=0}^{k} L_{k,j}(x) f(x_{n+1-j}), \tag{2}$$

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where $L_{k,j}(x)=\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{(x-x_{n+1-i})}{(x_{n+1-j}-x_{n+1-i})}$ for each $j=0,\ 1,\ \ldots,\ k.$ The resulting polynomial is given by

$$\begin{aligned}P(x)=&\frac{(x-x_{n-1})(x-x_n)(x-x_{n+1})(x-x_{n+2})}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)(x_{n-2}-x_{n+1})(x_{n-2}-x_{n+2})}y_{n-2}\\&+\frac{(x-x_{n-2})(x-x_n)(x-x_{n+1})(x-x_{n+2})}{(x_{n-1}-x_{n-2})(x_{n-1}-x_n)(x_{n-1}-x_{n+1})(x_{n-1}-x_{n+2})}y_{n-1}\\&+\frac{(x-x_{n-2})(x-x_{n-1})(x-x_{n+1})(x-x_{n+2})}{(x_n-x_{n-2})(x_n-x_{n-1})(x_n-x_{n+1})(x_n-x_{n+2})}y_n\\&+\frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+2})}{(x_{n+1}-x_{n-2})(x_{n+1}-x_{n-1})(x_{n+1}-x_n)(x_{n+1}-x_{n+2})}y_{n+1}\\&+\frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+1})}{(x_{n+2}-x_{n-2})(x_{n+2}-x_{n-1})(x_{n+2}-x_n)(x_{n+2}-x_{n+1})}y_{n+2},\end{aligned}\tag{3}$$

Replace $x=sh+x_{n+1}$ into (3) and differentiate once with respect to s. By substituting $s=0$ and $s=1$, the following equations are produced respectively:

$$P'(x_{n+1})=-\frac{1}{12}y_{n-2}+\frac{1}{2}y_{n-1}-\frac{3}{2}y_n+\frac{5}{6}y_{n+1}+\frac{1}{4}y_{n+2},\tag{4}$$

$$P'(x_{n+2})=\frac{1}{4}y_{n-2}-\frac{4}{3}y_{n-1}+3y_n-4y_{n+1}+\frac{25}{12}y_{n+2}.\tag{5}$$

Consider $hf_{n+1}=P'(x_{n+1})$ in (4) and $hf_{n+2}=P'(x_{n+2})$ in (5) to obtain the following equations:

$$\begin{aligned}y_{n+1}-\frac{1}{10}y_{n-2}+\frac{3}{5}y_{n-1}-\frac{9}{5}y_n+\frac{3}{10}y_{n+2}&=\frac{12}{10}hf_{n+1},\\ y_{n+2}+\frac{3}{25}y_{n-2}-\frac{16}{25}y_{n-1}+\frac{36}{25}y_n-\frac{48}{25}y_{n+1}&=\frac{12}{25}hf_{n+2}.\end{aligned}\tag{6}$$

The next formulation is based on the strategy discussed by Celaya and Anza (2013). Equations (6) are inserted with five independent parameters $\alpha,\ \beta,\ \rho,\ \mu$ and δ which can be expressed as follows:

$$\begin{aligned}&\frac{12}{10}\left[(1+\alpha)hf_{n+1}-\alpha f_n\right]=(1+\rho)y_{n+1}-\rho y_n\\&-\frac{1}{10}y_{n-2}+\frac{3}{5}\left[(1+\delta)y_{n-1}-\delta y_{n-2}\right]\\&-\frac{9}{5}\left[(1+\mu)y_n-\mu y_{n-1}\right]+\frac{3}{10}\left[(1+\beta)y_{n+2}-\beta y_{n+1}\right],\\[2ex]&\frac{12}{25}h\left[(1+\alpha)f_{n+2}-\alpha f_{n+1}\right]=(1+\beta)y_{n+2}-\beta y_{n+1}\\&+\frac{3}{25}y_{n-2}-\frac{16}{25}\left[(1+\delta)y_{n-1}-\delta y_{n-2}\right]\\&+\frac{36}{25}\left[(1+\mu)y_n-\mu y_{n-1}\right]-\frac{48}{25}\left[(1+\rho)y_{n+1}-\rho y_n\right]\end{aligned}\tag{7}$$

After some algebraic manipulations, equations (7) can be obtained as follows:

$$\begin{aligned}&\left(\frac{12}{10}+\frac{12}{10}\alpha\right)hf_{n+1}-\frac{12}{10}\alpha hf_n,=\left(\frac{3}{10}+\frac{3}{10}\beta\right)y_{n+2}\\&+\left(1+\rho-\frac{3}{10}\beta\right)y_{n+1}+\left(-\frac{9}{5}-\frac{9}{5}\mu-\rho\right)y_n\\&+\left(\frac{3}{5}+\frac{3}{5}\delta+\frac{9}{5}\mu\right)y_{n-1}+\left(-\frac{3}{5}\delta-\frac{1}{10}\right)y_{n-2},\\[2ex]&\left(\frac{12}{25}+\frac{12}{25}\alpha\right)hf_{n+2}-\frac{12}{25}\alpha hf_{n+1}=(1+\beta)y_{n+2}\\&+\left(-\beta-\frac{48}{25}-\frac{48}{25}\rho\right)y_{n+1}+\left(\frac{36}{25}+\frac{36}{25}\mu+\frac{48}{25}\rho\right)y_n\\&+\left(-\frac{16}{25}\delta-\frac{16}{25}-\frac{36}{25}\mu\right)y_{n-1}+\left(\frac{16}{25}\delta+\frac{3}{25}\right)y_{n-2}.\end{aligned}\tag{8}$$

Equations (8) can be written in the form of linear multistep method (LMM), $\sum_{j=0}^{4}A_jy_{n+j-2}=h\sum_{j=0}^{4}B_jf_{n+j-2}$ which is presented as follows:

$$\begin{aligned}&A_0y_{n-2}+A_1y_{n-1}+A_2y_n+A_3y_{n+1}+A_4y_{n+2}\\&=B_2hf_n+B_3hf_{n+1}+B_4hf_{n+2}.\end{aligned}\tag{9}$$

where

$$A_0=\begin{bmatrix}-\frac{3}{5}\delta-\frac{1}{10}\\ \frac{16}{25}\delta+\frac{3}{25}\end{bmatrix},\ A_1=\begin{bmatrix}\frac{3}{5}+\frac{3}{5}\delta+\frac{9}{5}\mu\\ -\frac{16}{25}\delta-\frac{16}{25}-\frac{36}{25}\mu\end{bmatrix},$$

$$A_2=\begin{bmatrix}-\frac{9}{5}-\frac{9}{5}\mu-\rho\\ \frac{36}{25}+\frac{36}{25}\mu+\frac{48}{25}\rho\end{bmatrix},\ A_3=\begin{bmatrix}1+\rho-\frac{3}{10}\beta\\ -\beta-\frac{48}{25}-\frac{48}{25}\rho\end{bmatrix},$$

$$A_4=\begin{bmatrix}\frac{3}{10}+\frac{3}{10}\beta\\ 1+\beta\end{bmatrix},\ B_2=\begin{bmatrix}-\frac{12}{10}\alpha\\ 0\end{bmatrix},$$

$$B_3=\begin{bmatrix}\frac{12}{10}+\frac{12}{10}\alpha\\ -\frac{12}{25}\alpha\end{bmatrix},\ B_4=\begin{bmatrix}0\\ \frac{12}{25}+\frac{12}{25}\alpha\end{bmatrix}.$$

III. ORDER OF THE METHOD

The method is said to be order p if $C_0=C_1=\ldots=C_p=0,\ C_{p+1}\neq 0$ where C_{p+1} is error constant (Lambert, 1991). Due to the involvement of several parameters during the process of the derivation, the order of the derived method must be determined so that the coefficients of derived formula will possess only one independent parameter α. The order of the formula which corresponds to (9) is proven as follows:

$$C_0=\sum_{j=0}^{4}A_j=\begin{bmatrix}0\\0\end{bmatrix},$$

$$C_1=\sum_{j=0}^{4}\left(jA_j-B_j\right)=\begin{bmatrix}\frac{3}{5}\delta+\rho-\frac{9}{5}\mu+\frac{3}{10}\beta\\ -\frac{16}{25}\delta-\frac{48}{25}\rho+\frac{36}{25}\mu+\beta\end{bmatrix},$$

The formula $C_q=\sum_{j=0}^{4}\left(\frac{1}{q!}j^qA_j-\frac{1}{(q-1)!}j^{q-1}B_j\right)$ where $q=2,\ 3,\ 4,\ 5$ is used to obtain the following constants:

$$C_2=\begin{bmatrix}\frac{3}{10}\delta+\frac{5}{2}\rho-\frac{27}{10}\mu+\frac{21}{20}\beta-\frac{6}{5}\alpha\\ -\frac{8}{25}\delta-\frac{24}{5}\rho+\frac{54}{25}\mu+\frac{7}{2}\beta-\frac{12}{25}\alpha\end{bmatrix},$$

$$C_3=\begin{bmatrix}\frac{1}{10}\delta+\frac{19}{6}\rho-\frac{21}{10}\mu+\frac{37}{20}\beta-3\alpha\\ -\frac{8}{75}\delta-\frac{152}{25}\rho+\frac{42}{25}\mu+\frac{37}{6}\beta-\frac{42}{25}\alpha\end{bmatrix},$$

$$C_4=\begin{bmatrix}\frac{1}{40}\delta+\frac{65}{24}\rho-\frac{9}{8}\mu+\frac{35}{16}\beta-\frac{19}{5}\alpha\\ -\frac{2}{75}\delta-\frac{26}{5}\rho+\frac{9}{10}\mu+\frac{175}{24}\beta-\frac{74}{25}\alpha\end{bmatrix},$$

$$C_5=\begin{bmatrix}\frac{1}{200}\delta+\frac{211}{120}\rho-\frac{93}{200}\mu+\frac{781}{400}\beta-\frac{13}{4}\alpha+\frac{3}{50}\\ -\frac{2}{375}\delta-\frac{422}{125}\rho+\frac{93}{250}\mu+\frac{781}{120}\beta-\frac{7}{2}\alpha-\frac{12}{125}\end{bmatrix}.$$

The method given will be an order 4 if all parameters, $\alpha,\ \beta,\ \rho,\ \mu$ and δ verify the following conditions:

For the first point, y_{n+1}:

$$\beta=\frac{4}{3}\alpha,\ \rho=\frac{3}{5}\alpha,\ \mu=\frac{2}{3}\alpha,\ \delta=\frac{1}{3}\alpha$$

For the second point, y_{n+2}:

$$\beta=\frac{22}{25}\alpha,\ \rho=\frac{3}{4}\alpha,\ \mu=\frac{1}{2}\alpha,\ \delta=\frac{1}{4}\alpha$$

The error constant is $C_5=\begin{bmatrix}\frac{1}{10}\alpha+\frac{3}{50}\\ -\frac{3}{25}\alpha-\frac{12}{125}\end{bmatrix}$.

Subsequently, all the conditions are substituted into equations (8) and leave α as the free parameter. Hence, the BBDF (α) can be obtained as follows:

$$\left(-\frac{1}{10}-\frac{1}{5}\alpha\right)y_{n-2}+\left(\frac{3}{5}+\frac{7}{5}\alpha\right)y_{n-1}+\left(-\frac{9}{5}-\frac{9}{5}\alpha\right)y_n$$
$$+\left(\frac{3}{10}+\frac{2}{5}\alpha\right)y_{n+2}+\left(1+\frac{1}{5}\alpha\right)y_{n+1}$$
$$=\left(\frac{12}{10}+\frac{12}{10}\alpha\right)hf_{n+1}-\frac{12}{10}\alpha hf_n,$$

$$\left(1+\frac{22}{25}\alpha\right)y_{n+2}+\left(\frac{3}{25}+\frac{4}{25}\alpha\right)y_{n-2}+\left(-\frac{16}{25}-\frac{22}{25}\alpha\right)y_{n-1}$$
$$+\left(\frac{36}{25}+\frac{54}{25}\alpha\right)y_n+\left(-\frac{48}{25}-\frac{58}{25}\alpha\right)y_{n+1}$$
$$=\left(\frac{12}{25}+\frac{12}{25}\alpha\right)hf_{n+2}-\frac{12}{25}\alpha hf_{n+1}. \quad (10)$$

IV. STABILITY ANALYSIS

The basic difficulty in the numerical solution of stiff systems is the satisfaction of the requirement of absolute stability. Based on Hall and Watt (1976), the LMM is A-stable if its region of absolute stability contains the whole of the left-hand half-plane, $R(h\lambda)<0$. The stability region for BBDF (α) can be obtained by applying equations (10) to the standard linear equation, $f=\lambda y$ which takes the form:

$$\left(\frac{1}{10}+\frac{1}{5}\alpha\right)y_{n-2}+\left(-\frac{3}{5}-\frac{7}{5}\alpha\right)y_{n-1}$$
$$+\left(\frac{9}{5}+\frac{9}{5}\alpha-\frac{12}{10}\alpha h\lambda\right)y_n=\left(\frac{3}{10}+\frac{2}{5}\alpha\right)y_{n+2}$$
$$+\left(1+\frac{1}{5}\alpha-\frac{12}{10}h\lambda-\frac{12}{10}\alpha h\lambda\right)y_{n+1},$$

$$\left(-\frac{3}{25}-\frac{4}{25}\alpha\right)y_{n-2}+\left(\frac{16}{25}+\frac{22}{25}\alpha\right)y_{n-1}$$
$$+\left(-\frac{36}{25}-\frac{54}{25}\alpha\right)y_n=\left(-\frac{48}{25}-\frac{58}{25}\alpha+\frac{12}{25}\alpha h\lambda\right)y_{n+1}$$
$$+\left(1+\frac{22}{25}\alpha-\frac{12}{25}h\lambda-\frac{12}{25}\alpha h\lambda\right)y_{n+2}. \quad (11)$$

Consider $\hat{h}=h\lambda$, equations (11) can be written in the matrix form:

$$A\begin{bmatrix}y_{n+1}\\y_{n+2}\end{bmatrix}=B\begin{bmatrix}y_{n-1}\\y_n\end{bmatrix}+C\begin{bmatrix}y_{n-3}\\y_{n-2}\end{bmatrix}. \quad (12)$$

where

$$A=\begin{bmatrix}1+\frac{1}{5}\alpha-\frac{12}{10}\hat{h}-\frac{12}{10}\alpha\hat{h} & \frac{3}{10}+\frac{2}{5}\alpha\\ -\frac{48}{25}-\frac{58}{25}\alpha+\frac{12}{25}\alpha\hat{h} & 1+\frac{22}{25}\alpha-\frac{12}{25}\hat{h}-\frac{12}{25}\alpha\hat{h}\end{bmatrix},$$

$$B=\begin{bmatrix}-\frac{3}{5}-\frac{7}{5}\alpha & \frac{9}{5}+\frac{9}{5}\alpha-\frac{12}{10}\alpha\hat{h}\\ \frac{16}{25}+\frac{22}{25}\alpha & -\frac{36}{25}-\frac{54}{25}\alpha\end{bmatrix},$$

$$C=\begin{bmatrix}0 & \frac{1}{10}+\frac{1}{5}\alpha\\ 0 & -\frac{3}{25}-\frac{4}{25}\alpha\end{bmatrix}.$$

By solving $\det\left(At^2-Bt-C\right)$, the stability polynomial, $p\left(t,\hat{h},\alpha\right)$ for (10) is produced as follows:

$$p\left(t,\hat{h},\alpha\right)=\frac{12}{25}t^2\hat{h}\alpha+\frac{1}{25}t-\frac{9}{25}t^2+\frac{18}{125}t^2\alpha$$
$$-\frac{18}{125}t^2\hat{h}-\frac{372}{125}t^4\alpha\hat{h}-\frac{168}{125}t^4\alpha^2\hat{h}$$
$$+\frac{144}{125}t^4\hat{h}^2\alpha-\frac{264}{125}t^3\hat{h}\alpha+\frac{72}{125}t^4\hat{h}^2\alpha^2$$
$$+\frac{48}{125}t^3\alpha^2\hat{h}+\frac{24}{25}\alpha^2t^2\hat{h}-\frac{72}{125}t^3\hat{h}^2\alpha^2$$
$$+\frac{6}{125}at+\frac{318}{125}at^4-\frac{42}{25}\hat{h}t^4-\frac{342}{125}\alpha t^3$$
$$+\frac{138}{125}\alpha^2t^4-\frac{54}{25}\alpha^2t^3+\frac{126}{125}\alpha^2t^2+\frac{72}{125}\hat{h}^2t^4$$
$$-\frac{252}{125}t^3\hat{h}+\frac{6}{125}\alpha^2t+\frac{197}{125}t^4-\frac{153}{125}t^3. \quad (13)$$

The stability region is the region enclosed by the set of points for which $|t|=1$ where the boundary of the stability region can be mapped out by substituting $t=e^{i\theta}$, $0\le\theta\le 2\pi$ into the stability polynomial (13). The graph of stability region is plotted using MAPLE software.

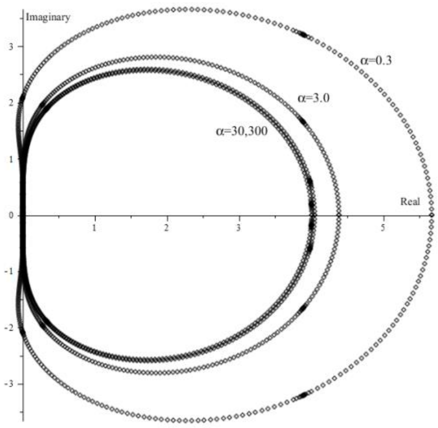

Figure 1. Stability regions of BBDF (α)

Figure 1 presents the graph of stability region for BBDF (α) with different values of parameter, $\alpha = 0.3,\ 3,\ 30,\ 300$. It has to be noted that the stability regions cover almost the entire negative half plane. Thus, the BBDF (α) is A-stable.

V. IMPLEMENTATION OF THE METHOD

This section deals with the implementation of the BBDF (α)using Newton iteration. The following notation is introduced to specify the notation:

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)},\ j = 1,\ 2. \tag{14}$$

where the notation i is introduced to specify the iteration, the $y_{n+j}^{(i+1)}$ denotes the $(i+1)^{\text{th}}$ iteration values of y_{n+j} and $e_{n+j}^{(i+1)}$ denotes the differences between $(i)^{\text{th}}$ and $(i+1)^{\text{th}}$ iteration values of y_{n+j}. The approximated values of y_{n+j} are computed from $y_{n+j}^{(i+1)} = y_{n+j}^{(i)} + e_{n+j}^{(i+1)}$. Therefore, the Newton's iteration takes the form:

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - F_j{}'\left(y_{n+j}^{(i)}\right)^{-1} F_j\left(y_{n+j}^{(i)}\right),\ j = 1,\ 2. \tag{15}$$

which can be written as

$$F_j'\left(y_{n+j}^{(i)}\right) e_{n+j}^{(i+1)} = -F_j\left(y_{n+j}^{(i)}\right).$$

The absolute error is $\text{error}^{(i+1)} = \left| y_{\text{exact}}^{(i+1)} - y_{\text{approximate}}^{(i+1)} \right|$ while the maximum error is $\text{MAXE} = \max\left[\text{error}^{(i+1)}\right]$.

VI. NUMERICAL EXPERIMENTS

In this section, the values $\alpha = 0.3,\ 3,\ 30,\ 300$ are selected for the numerical computation due to their A-stable properties. Note that the problems and results for existing methods are taken from Abasi*et al.* (2014). The graphs of Log (MAXE) against Log(h) are illustrated in Figures 2-3. The following notations are used in the tables and figures:

h	: Step size.
α	: Independent parameter.
MAXE	: Maximum error.
BBDF (5)	: 2-point BBDF of order five (Nasir *et al.*, 2012).
2OBBDF (5)	: 2 off-step points BBDF of order five (Abasi*et al.*, 2014).
BBDF (α)	: Block backward differentiation formulas (α) of order four.

Problem 1:

$$y' = -20y + 20\sin x + \cos x,\ y(0) = 1,\ 0 \le x \le 2.$$

The exact solution for this problem is given by

$$y(x) = \sin x + e^{-20x}.$$

Eigenvalue: -20.

Problem 2:

$$y'_1 = 32y_1 + 66y_2 + \frac{2}{3}x + \frac{2}{3},$$
$$y'_2 = -66y_1 - 133y_2 - \frac{1}{3}x - \frac{1}{3},\ 0 \le x \le 1.$$

The exact solutions are

$$y_1(0) = \frac{1}{3},\ y_1(x) = \frac{2}{3}x + \frac{2}{3}e^{-x} - \frac{1}{3}e^{-100x},$$
$$y_2(0) = \frac{1}{3},\ y_2(x) = -\frac{1}{3}x - \frac{1}{3}e^{-x} + \frac{2}{3}e^{-100x}.$$

Eigenvalues: -1, -100.

Table 1. Numerical results of Problem 1

h	Methods	α	MAXE
10^{-2}	BBDF(5)	-	8.85478E-2
	2OBBDF(5)	-	8.05923E-2
	BBDF(α)	0.3	3.66822E-2
		3	3.98408E-2
		30	4.34192E-2
		300	4.83403E-2
10^{-4}	BBDF(5)	-	1.46428E-3
	2OBBDF(5)	-	1.46355E-3
	BBDF(α)	0.3	8.91419E-6
		3	1.37939E-5
		30	5.66628E-5
		300	2.80852E-4
10^{-6}	BBDF(5)	-	1.47126E-5
	2OBBDF(5)	-	1.47126E-5
	BBDF(α)	0.3	9.00713E-10
		3	1.43375E-9
		30	6.80402E-9
		300	5.90049E-8

Table 2. Numerical results of Problem 2

h	Methods	α	MAXE
10^{-2}	BBDF(5)	-	1.21580E-2
	2OBBDF(5)	-	1.20347E-2
	BBDF(α)	0.3	4.42072E-3
		3	4.41510E-3
		30	4.41245E-3
		300	4.41209E-3
10^{-4}	BBDF(5)	-	4.78743E-3
	2OBBDF(5)	-	4.77571E-3
	BBDF(α)	0.3	1.42482E-4
		3	2.38160E-4
		30	2.35272E-3
		300	2.25767E-2
10^{-6}	BBDF(5)	-	4.90322E-5
	2OBBDF(5)	-	4.90310E-5
	BBDF(α)	0.3	1.50048E-8
		3	2.55771E-8
		30	2.58140E-7
		300	2.61435E-6

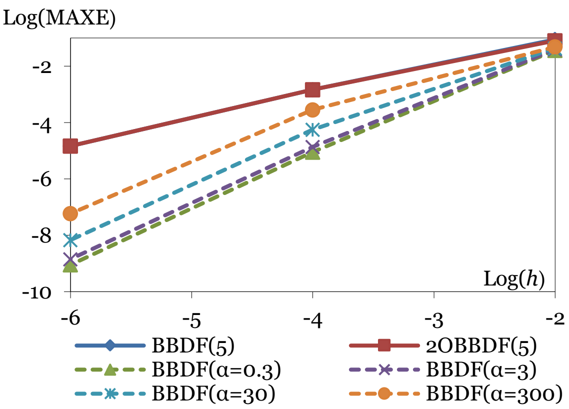

Figure 2. Accuracy curves for Problem 1

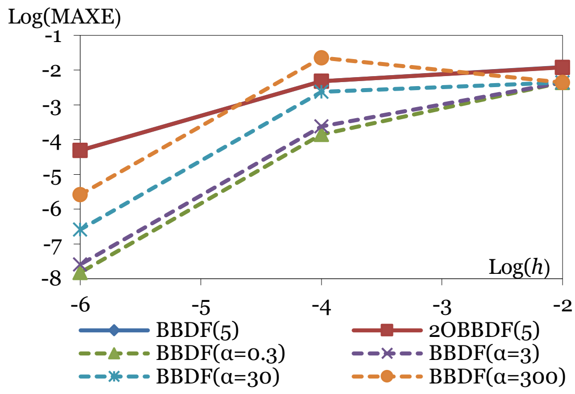

Figure 3. Accuracy curves for Problem 2

VII. DISCUSSIONS

In table 1, it is observed that the similar level of accuracy is obtained by BBDF (α), BBDF (5) and 2OBBDF (5) at $h = 10^{-2}$. However, the BBDF (α) show a significant improvement in accuracy when compared to the BBDF (5) and 2OBBDF (5) as the step size decreases. Although the BBDF (α) has one order less than the BBDF (5) and 2OBBDF (5), the derived method manages to outperform both existing methods in terms of maximum error at most of h. From all tables given, it can be seen clearly that the accuracy of BBDF (α) deteriorates when the values of α increases due to the value of error constants.

VIII. CONCLUSIONS

Overall, the existence of the independent parameter α in the derived method influences the approximation, hence gives better accuracy than the existing block methods. Therefore, the BBDF (α) can be applied as an alternative stiff ODEs solver.

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