The Exterior Square of a Bieberbach Group with Quaternion Point Group of Order Eight

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A Bieberbach group is defined to be a torsion free crystallographic group which is an extension of a free abelian lattice group by a finite point group. This paper aims to determine a mathematical representation of a Bieberbach group with quaternion point group of order eight. Such mathematical representation is the exterior square. Mathematical method from representation theory is used to find the exterior square of this group. The exterior square of this group is found to be nonabelian.

Keywords: mathematical structure; exterior square; Bieberbach group; quaternion point group

I. INTRODUCTION

Mathematical method from representation theory has been one of the important keys in the study of the structures of a crystal. Bieberbach group is a crystallographic group. In this research, the mathematical representation of a Bieberbach group of dimensions six with quaternion point group of order eight, denoted as $Q_4(6)$ is computed. Such mathematical representation is the exterior square of $Q_4(6)$, denoted as $Q_4(6) \wedge Q_4(6)$. In order to compute the mathematical representation of the group, various mathematical approaches have been introduced based on its property.

In 1987, by using the groups’ presentation, the nonabelian tensor square of all small groups of order up to 30 were found (Brown et. al., 1987). Meanwhile, the nonabelian tensor squares of 2-generator 2-groups of class 2 using the crossed pairing method were computed in 1999 (Kappe et. al., 1999). The technique of constructing a group $\nu(G)$ and proving that its subgroup $[G, G^\phi]$ is isomorphic to the nonabelian tensor square of a group $G$ has been developed (Rocco, 1991), but, due to the limitations of the usage of crossed pairing method, (Blyth & Morse, 2009) extended the method involving $\nu(G)$ to compute the nonabelian tensor squares of the polycyclic groups. A research on the mathematical representation of infinite nonabelian 2-generator groups of nilpotency class two has been conducted by using the classification and the nonabelian tensor squares of the groups (Mohd Ali et. al., 2007). Method developed by Blyth and Morse has been used to compute the nonabelian tensor square of Bieberbach groups with abelian point group (Masri, 2009) and also Bieberbach groups with nonabelian point group (Mohd Idrus, 2011).

The objective of this research is to find the mathematical representation of a group which is the exterior square of a Bieberbach group with quaternion point group of order eight.

II. METHODS

To find the mathematical representations of a group, the method involving a group $\nu(G)$ developed by (Rocco, 1991) is used. The group $\nu(G)$ is defined as follows:

Definition 1 (Rocco, 1991)
Let $G$ be a group with presentation $\langle G \mid R \rangle$ and let $G^\varphi$ be an isomorphic copy of $G$ via the mapping $\varphi : g \to g^\varphi$ for all $g \in G$. The group $\nu(G)$ is defined to be

$$\nu(G) = \{G, G^\varphi \mid R, R^\varphi, [g, h^\varphi] = [g^\varphi, (h^\varphi)^\varphi]\}
= \{g, h^\varphi\}^{\varphi^{-1}}, \forall x, g, h \in G\}
$$

where $g^h = h^{-1}gh$ and $[g, h] = g^{-1}h^g - h$. The important fact about $\nu(G)$ is that its subgroup $[G, G^\varphi]$ is actually isomorphic to the nonabelian tensor square of the group $G$. An analysis of the group $\nu(G)$ for arbitrary finite and infinite groups $G$ as a tool to compute $G \otimes G$ and other mathematical representations has been provided (Blyth & Morse, 2009). Furthermore, the nonabelian tensor square of a polycyclic group given by a polycyclic presentation can be computed (Eick & Nickel, 2008). Moreover, if $G$ is polycyclic, then $G \otimes G$ is polycyclic. Hence, $G \otimes G$ has a consistent polycyclic presentation (Blyth & Morse, 2009). To show that the group is polycyclic, the polycyclic presentation has to be consistent. Thus, the following two definitions are needed.

**Definition 2** (Eick & Nickel, 2008) Polycyclic Presentation

Let $F_n$ be a free group on generators $g_1, ..., g_n$ and $R$ be a set of relations of group $F_n$. The relations of a polycyclic presentation $F_n/\langle R \rangle$ have the form:

$$g_i^\epsilon_i = g_i^{y_{i,1}} \cdots g_i^{y_{i,n}} \quad \text{for } i \in I,$$
$$g_j^{-1} g_i g_j = g_j^{y_{j,1}} \cdots g_j^{y_{j,n}} \quad \text{for } j < I,$$
$$g_j g_i g_j^{-1} = g_j^{z_{j,1}} \cdots g_j^{z_{j,n}} \quad \text{for } j < I \text{ and } j \notin I$$

for some $I \subseteq \{1, ..., n\}$, certain exponents $\epsilon_i \in \mathbb{Z}$ for $i \in I$
and $y_{i,j}, y_{j,k}, z_{i,j,k} \in \mathbb{Z}$ for all $i, j$ and $k$.

**Definition 3** (Eick & Nickel, 2008) Consistent Polycyclic Presentation

Let $G$ be a group on generated by $g_1, ..., g_n$ and the consistency relations in $G$ can be evaluated in the polycyclic presentation of $G$ using the collection from the left as in the following:

$$g_k(g_j g_i) = (g_k g_j) g_i \quad \text{for } k > j > i,$$
$$g_j^\epsilon_i g_i = g_j^{\epsilon_i - 1} g_j g_i \quad \text{for } j > i, j \in I,$$
$$g_j^\epsilon_i g_i = (g_j g_i) g_i^{\epsilon_i - 1} \quad \text{for } j > i, i \in I,$$
$$g_j = (g_j g_i)^{-1} g_i \quad \text{for } j > i, i \notin I$$

for some $I \subseteq \{1, ..., n\}$, $\epsilon_i \in \mathbb{Z}$. Then, $G$ is said to be given by a consistent polycyclic presentation.

A Bieberbach group of dimensions six with quaternion point group of order eight that has been considered is isomorphic to the group below:

$$G = \langle a_0, a_1, l_1, l_2, l_1 l_1 l_2, i_6 \rangle \quad \text{where}$$
$$a_0 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \quad \text{, (1)}$$
polycyclic presentation is found to be
\[ Q_1(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \rangle \]
\[ a^2 = c l_6, \quad b^2 = c l_6^{-1}, \quad c^2 = l_1^6 l_6^{-1}, \]
\[ b^n = b c^{-1} l_6^{-1}, \quad c^n = c, \quad c^b = c, \]
\[ l_1^n = l_1, \quad l_2^n = l_2, \quad l_3^n = l_3^{-1}, \]
\[ l_4^n = l_4, \quad l_5^n = l_5, \quad l_6^n = l_6^{-1}, \]
\[ l^n_j = l_j, \quad l^n_j = l_j \quad \text{for } j > i, 1 \leq i, j \leq 6 \}.

Then, \( Q_1(6) \) is consistent.

To find the mathematical representation which is the exterior square of a group, \( G \wedge G \), the next theorem that indicates \( G \wedge G \) is isomorphic to \( [G,G^\circ]_{\wedge(G)} \) is stated.

Before that, the following definition is needed.

**Theorem 2**

Let \( Q_1(6) \) be a Bieberbach group of dimensions six with quaternion point group of order eight. Then, the central subgroup of \( Q_1(6) \), denoted as \( V(Q_1(6)) \) is given as:

\[ V(Q_1(6)) = \langle a \otimes a, b \otimes b, l_i \otimes l_i, (a \otimes l_i)(l_i \otimes a), \]
\[ (b \otimes l_i)(l_i \otimes b) \rangle. \]

**Definition 4** (Blyth & Morse, 2009)

Let \( G \) be any group. Then \( \tau(G) \) is defined to be quotient group \( \nu(G)/[\sigma(\nu(G))] \), where

\[ \sigma : G \otimes G \rightarrow [G,G^\circ]_{\wedge(G)} \]

defined by \( \sigma (g \otimes h) = [g,h^\circ] \) for all \( g,h \) in \( G \) is an isomorphism.
Theorem 3 (Blyth & Morse, 2009)

Let $G$ be any group. The map 

$$
\hat{\sigma} : G \times G \to \left[ G, G^\omega \right]_{\tau(G)} \ni \tau(G)
$$

defined by

$$
\hat{\sigma}(g, h) = \left[ g, h^\omega \right]_{\tau(G)}
$$

is an isomorphism.

Hence, in this research, $\left[ G, G^\omega \right]_{\tau(G)}$ is computed in order to find the exterior square of a group. Since $\tau(G)$ is a subgroup of $\left[ g, h^\omega \right]_{\tau(G)}$, coincides with $\left[ g, h^\omega \right]$. Therefore, for simplification, $\left[ g, h^\omega \right]$ is used instead of $\left[ g, h^\omega \right]_{\tau(G)}$.

Theorem 4 gives a complete description of the generators of $\left[ G, G^\omega \right]$ and $\left[ G, G^\omega \right]_{\tau(G)}$ in terms of a polycyclic generating set of $G$. This theorem is used in computing the nonabelian tensor square and the exterior square of a group $G$.

Theorem 4 (Blyth & Morse, 2009)

Let $G$ be a polycyclic group with a polycyclic generating sequence $g_1, \ldots, g_k$. Then $\left[ G, G^\omega \right]$ a subgroup of $\nu(G)$ is given by

$$
\left[ G, G^\omega \right] = \left\{ \left[ g, g_i^\omega \right], \left[ g_i, (g^\omega)^e \right], \left[ g, g_j^\omega \right], \left[ g_j, g_i^\omega \right] \right\}
$$

and

$$\left[ G, G^\omega \right]_{\tau(G)} \ni \tau(G)$$

a subgroup of $\tau(G)$ is given by

$$
\left[ G, G^\omega \right]_{\tau(G)} = \left\{ \left[ g_i^\omega, (g_j^\omega)^e \right], \left[ g_j, (g_i^\omega)^\delta \right] \right\}
$$

for $1 \leq i < j \leq k$, where

$$
\varepsilon = \begin{cases} 
1, & \text{if } |g_i| < \infty, \\
\pm 1, & \text{if } |g_i| = \infty,
\end{cases}
$$

and

$$
\delta = \begin{cases} 
1, & \text{if } |g_j| < \infty, \\
\pm 1, & \text{if } |g_j| = \infty.
\end{cases}
$$

III. RESULTS AND DISCUSSION

A mathematical representation, namely the exterior square of $Q_1(6) \wedge Q_1(6)$, is presented in this section. As mentioned in Section II, the subgroup of $\nu(G)$, $\left[ G, G^\omega \right]_{\tau(G)}$ is isomorphic to the exterior square of a group, therefore the exterior square can be computed by finding its $\left[ Q_1(6), Q_1(6) \right]_{\tau(G)}$. Thus, the exterior square of $Q_1(6)$ is presented as follows:

Theorem 5

The exterior square of $Q_1(6)$ is nonabelian and is given as

$$
Q_1(6) \wedge Q_1(6) = \left\{ a \wedge b, a \wedge c, c \wedge a, a \wedge l_1, a \wedge l_2, l_2 \wedge a, a \wedge l_6, b \wedge l_1, b \wedge l_2, l_2 \wedge b, b \wedge l_3, l_3 \wedge b, b \wedge l_6, c \wedge l_2, c \wedge l_6 \right\}
$$

Proof

The exterior square of $Q_1(6)$ is isomorphic to $\left[ Q_1(6), Q_1(6) \right]_{\tau(G)}$ by Theorem 3. Then, based on Theorem 4, $\left[ Q_1(6), Q_1(6) \right]_{\tau(G)}$ is generated by
\[
\left[ Q_5(6) , Q_6(6) \right]_{\mathcal{R}(G)} = \\
\left[ [ a^{11} , b^{11} ] , [ b^{12} , a^{21} ] , [ a^{21} , c^{12} ] , [ c^{11} , a^{21} ] , [ a^{21} , l_1^{12} ] , [ l_1^{11} , a^{12} ] , [ a^{12} , l_2^{11} ] , [ l_2^{12} , a^{11} ] , [ a^{11} , l_3^{12} ] , [ l_3^{11} , a^{22} ] , [ a^{22} , l_4^{11} ] , [ l_4^{12} , l_5^{11} ] , [ l_5^{12} , a^{11} ] , [ a^{11} , l_6^{12} ] , [ l_6^{11} , a^{22} ] , [ b^{11} , c^{12} ] , [ c^{11} , b^{12} ] , [ b^{12} , l_1^{11} ] , [ l_1^{12} , b^{11} ] , [ b^{11} , l_2^{12} ] , [ l_2^{11} , b^{12} ] , [ b^{12} , l_3^{11} ] , [ l_3^{12} , b^{11} ] , [ b^{11} , l_4^{12} ] , [ l_4^{11} , b^{12} ] , [ b^{12} , l_5^{11} ] , [ l_5^{12} , b^{11} ] , [ b^{11} , l_6^{12} ] , [ l_6^{11} , b^{12} ] , [ c^{11} , l_2^{11} ] , [ l_2^{12} , c^{11} ] , [ c^{11} , l_3^{12} ] , [ l_3^{11} , c^{12} ] , [ c^{11} , l_4^{11} ] , [ l_4^{12} , c^{12} ] , [ c^{11} , l_5^{12} ] , [ l_5^{11} , c^{12} ] , [ c^{11} , l_6^{11} ] , [ l_6^{12} , c^{12} ] , [ l_1^{11} , l_2^{12} ] , [ l_2^{11} , l_1^{12} ] , [ l_1^{12} , l_2^{11} ] , [ l_2^{11} , l_3^{12} ] , [ l_3^{11} , l_2^{12} ] , [ l_2^{12} , l_3^{11} ] , [ l_1^{11} , l_4^{12} ] , [ l_4^{11} , l_1^{12} ] , [ l_1^{12} , l_4^{11} ] , [ l_4^{11} , l_5^{12} ] , [ l_5^{11} , l_4^{12} ] , [ l_4^{12} , l_5^{11} ] , [ l_1^{11} , l_6^{12} ] , [ l_6^{11} , l_1^{12} ] , [ l_1^{12} , l_6^{11} ] , [ l_6^{11} , l_5^{12} ] , [ l_5^{11} , l_6^{12} ] , [ l_2^{11} , l_3^{12} ] , [ l_3^{11} , l_2^{12} ] , [ l_2^{12} , l_3^{11} ] , [ l_3^{11} , l_4^{12} ] , [ l_4^{11} , l_3^{12} ] , [ l_3^{12} , l_4^{11} ] , [ l_1^{11} , l_5^{12} ] , [ l_5^{11} , l_1^{12} ] , [ l_1^{12} , l_5^{11} ] , [ l_5^{11} , l_6^{12} ] , [ l_6^{11} , l_5^{12} ] , [ l_5^{12} , l_6^{11} ] , [ l_2^{11} , l_4^{12} ] , [ l_4^{11} , l_2^{12} ] , [ l_2^{12} , l_4^{11} ] , [ l_4^{11} , l_6^{12} ] , [ l_6^{11} , l_4^{12} ] , [ l_4^{12} , l_6^{11} ] , [ l_1^{11} , l_6^{12} ] , [ l_6^{11} , l_1^{12} ] , [ l_1^{12} , l_6^{11} ] , [ l_6^{11} , l_5^{12} ] , [ l_5^{11} , l_6^{12} ] \right]
\]

For examples,
\[
\left[ a^{-1} , b^{12} \right] = \left[ a^{-1} , [a,b]^{12} \right]^{-1} \\
= \left[ a^{-1} , (c_{l_4}^{12})^{12} [a,b]^{12} \right]^{-1} \\
= \left[ a^{-1} , l_{l_4}^{12} [a^{-1},c]^{12} [a^{-1},c], [l_{l_4}^{12}]^{12} [a,b]^{12} \right]^{-1} \\
= \left[ a^{-1} , l_{l_4}^{12} [a^{-1},c]^{12} [a,b]^{12} \right]^{-1} \\
= \left[ [a,a^{12}]^{12} [a,c]^{12} [a,b]^{12} \right]^{-1} \\
= \left[ a,a^{12} \right]^{-1} [a,b]^{12} \\
= [a,b]^{12}.
\]

\[
\left[ b^{12} , b^{12} \right] = [b^{-1} , [a,b]^{12} [a,b]^{12}]^{-1} \\
= [b^{-1} , (c_{l_3}^{12})^{12} [a,b]^{12}]^{-1} \\
= [b^{-1} , l_{l_3}^{12} (b^{-1},c)^{12} (b^{-1},c), (l_{l_3}^{12})^{12} [a,b]^{12}]^{-1} \\
= [b^{-1} , l_{l_3}^{12} [b^{-1},c]^{12} [a,b]^{12}]^{-1} \\
= [b,r_{l_3}^{12} [b,c]^{12} [a,b]^{12}]^{-1} \\
= [b,l_{l_3}^{12} [b,c]^{12} [a,b]^{12}]^{-1} \\
= [b,l_{l_3}^{12} [b,c]^{12} [a,b]^{12}]^{-1} \\
= [b,l_{l_3}^{12} [b,c]^{12} [a,b]^{12}]^{-1} \\
= [a,b]^{12}.
\]

\[
\left[ a^{-1} , b^{-1} \right] = \left[ a^{-1} , [a,b]^{-1} \right]^{-1} \\
= \left[ a^{-1} , (c_{l_3}^{12})^{-1} [a,b]^{-1} \right]^{-1} \\
= \left[ a^{-1} , l_{l_3}^{12} [a^{-1},c]^{-1} [(a^{-1},c), (l_{l_3}^{12})^{-1} [a,b]^{-1} \right]^{-1} \\
= \left[ a^{-1} , l_{l_3}^{12} [a^{-1},c]^{-1} [a,b]^{-1} \right]^{-1} \\
= \left[ [a,a^{12}]^{-1} [a,c]^{-1} [a,b]^{-1} \right]^{-1} \\
= \left[ a,a^{12} \right]^{-1} [a,c]^{-1} [a,b]^{-1} \\
= [a,a^{12}]^{-1} [a,c]^{-1} [a,b]^{-1} \\
= [a,b]^{-1}.
\]

\[
\left[ a^{12} , b^{12} \right] = [a^{12} , [a,b]^{12} [a,b]^{12}]^{-1} \\
= [a^{12} , (c_{l_3}^{12})^{12} [a,b]^{12}]^{-1} \\
= [a^{12} , l_{l_3}^{12} [a^{12},c]^{12} [a^{12},c], [l_{l_3}^{12}]^{12} [a,b]^{12}]^{-1} \\
= [a^{12} , l_{l_3}^{12} [a^{12},c]^{12} [a,b]^{12}]^{-1} \\
= [a,a^{12}]^{-1} [a,c]^{-1} [a,b]^{-1} \\
= [a,a^{12}]^{-1} [a,c]^{-1} [a,b]^{-1} \\
= [a,a^{12}]^{-1} [a,c]^{-1} [a,b]^{-1} \\
= [a,a^{12}]^{-1} [a,c]^{-1} [a,b]^{-1} \\
= [a,b]^{-1}.
\]

However, some of the generators can be eliminated since some of them can be written in terms of other generators.

First, note that all elements in \( \nabla( Q_5(6) ) \) are trivial in \( Q_5(6) \land Q_6(6) \). Then, by Theorem 2 and 3, \([a,a^a],[b,b^b],[l_1,l_1^a],[a,b^b] \) are trivial in \( Q_5(6),Q_6(6) \), and \([a,l_1^a],[l_1,a^a] \) are trivial in \( Q_5(6),Q_6(6) \). By using the relations of \( Q_5(6) \) in (2) and some properties of the commutator calculus, the following results are obtained.
\([b^{-1}, a^\varphi] = [b^{-1}, [a, a^\varphi]] [b, a^\varphi]^{-1}\)
\(= [b^{-1}, (c^{-1} l_5^{-1})^\varphi] [b, a^\varphi]^{-1}\)
\(= [b^{-1}, l_5^{-\varphi}] [b^{-1}, c^{-\varphi}] [a^{-1}, c^{-1}] (l_5^{-1})^{-\varphi} [b, a^\varphi]^{-1}\)
\(= [b^{-1}, l_5^{-\varphi}] [b^{-1}, c^{-\varphi}] [b, a^\varphi]^{-1}\)
\(= [b, l_5^{-\varphi}] [b, a^\varphi]^{-1}\)
\(= [b, l_5^{-\varphi}] [b, b^\varphi]^2 [a, b^\varphi]\)
\(= [b, l_5^{-\varphi}] [a, b^\varphi]\).\]

\([b, a^\varphi] = [a^{-1}, [b, a^\varphi]] [b, a^\varphi]^{-1}\)
\(= [a^{-1}, (c^{-1} l_5^{-1})^\varphi] [b, a^\varphi]^{-1}\)
\(= [a^{-1}, l_5^{-\varphi}] [a^{-1}, c^{-\varphi}] [a^{-1}, c^{-1}] (l_5^{-1})^{-\varphi} [b, a^\varphi]^{-1}\)
\(= [a^{-1}, l_5^{-\varphi}] [a^{-1}, c^{-\varphi}] [b, a^\varphi]^{-1}\)
\(= [a, c^{-\varphi}] [a, c^{-\varphi}] [b, a^\varphi]^{-1}\)
\(= [a, b^\varphi]\).\]

\([b^{-1}, a^\varphi] = [b^{-1}, [b, a^\varphi]] [b, a^\varphi]^{-1}\)
\(= [b^{-1}, (l_5^{-1} c^{-1})^\varphi] [b, a^\varphi]\)
\(= [b^{-1}, l_5^{-\varphi}] [b, a^\varphi]\)
\(= [b^{-1}, l_5^{-\varphi}] [b, c^\varphi] [b, a^\varphi]\)
\(= [b, l_5^{-\varphi}] [b, c^\varphi] [b, a^\varphi]^{-1}\)
\(= (b, l_5^{-\varphi}) [b, b^\varphi]^2 [a, b^\varphi]^{-1}\)
\(= [b, l_5^{-\varphi}] [a, b^\varphi]^{-1}\).

By the relations of \(Q_i(6)\) in (2) \(a, b, l_2\) and \(l_6\) commute with \(c\). Moreover, \(l_5\) and \(l_6\) also commute with \(a\). Then,

\(a^\varphi = [a, c^\varphi] = [a, a^\varphi]^{-1}\),
\(c^\varphi = [c, a^\varphi]^{-1}\),
\(a^\varphi = [a, l_6^\varphi] = [a, l_6^\varphi]^{-1}\).

Also, \(l_i\) commutes with \(l_j\) for all \(1 \leq i < j \leq 6\),

\(l_i^{-\varphi} = [l_i, l_j^{-1}]^{-1}\),
\(l_i^{-\varphi} = [l_i, l_j^{-1}]^{-1}\),
\(l_i^{-\varphi} = [l_i, l_j^{-1}]^{-1}\).

Since \(l_1 = [l_i, l_j]^{-1}\) and \(l_1 = [l_i, l_j]^{-1}\) for all \(1 \leq i < j \leq 6\), the remaining generators of \(Q_6(6), Q_6(6)\) are

\(Q_6(6), Q_6(6) = [a, b^\varphi], [a, c^\varphi], [c, a^\varphi], [a, l_6^\varphi], [b, l_6^\varphi], [l_2, l_6^\varphi], [b, b^\varphi], [l_2, b^\varphi], [l_5, b^\varphi], [b, l_5^\varphi], [c, l_5^\varphi] \) and \(c, l_6^\varphi\). Then, by Theorem 3, \(Q_6(6) = Q_6(6)\) is generated by

\(a \land b \land a \land c \land a \land l_1, a \land l_2, l_2 \land a, a \land l_5, b \land l_1, b \land l_2 \land b \land l_5, l_5 \land b, b \land l_6, c \land l_2\) and \(c \land l_6\). The
next step is to show that $Q_1(6) \wedge Q_1(6)$ is nonabelian.

Since,
\[
[[a,b^s][a,l_1^s]] = [c, (l_1^{-1})^{-s}]
\]
\[
= [c, (l_1^{-1})^{-s}][c, (l_1^{-1})^{-s}][l_1^{-1}][l_1^{-s}][l_1^{-1}]
\]
\[
= [c, (l_1^{-1})^{-s}][l_1^{-2}][l_1^{-s}][l_1^{-1}]
\]
\[
= [c, (l_1^{-1})^{-s}]
\]
\[
= [c, l_1^{-s}][c, l_1^{-s}]
\]
\[
= [l_1^{-s}][l_1^{-s}]
\]
\[
= [l_1^{-s}][l_1^{-s}]
\]
\[
\neq 1.
\]

Then, $[Q_1(6) \wedge Q_1(6)]_{(1)}$ is nonabelian which implies $Q_1(6) \wedge Q_1(6)$ to be nonabelian.

**IV. SUMMARY**

In this paper, the exterior square of a Bieberbach group with quaternion point group of order eight is computed. The result shows that the exterior square of this group is nonabelian.

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**VI. REFERENCES**


