Homotopy Perturbation Method for Boundary Value Problems with Delay Differential Equations

Hamood. M. Yousef* and A. I. B. MD. Ismail

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Pulau Pinang

This study uses the Homotopy Perturbation Method (HPM) to solve the second order of the boundary value problem (BVP) of Delay Differential Equations (DDEs). The aim of this paper to obtain the approximate solution for BVP by using the HPM with Pade approximation. Then compare the results obtained with the exact solution. A set of examples to illustrate the efficiency of the method. HPM provides an approximate analytical solution in the form of a convergent series.

Keywords: Delay Differential Equations, Boundary Value Problems, Homotopy Perturbation Method, Approximate solution, Padé approximation, Convergence of Method

I. INTRODUCTION

Recently, there are many problems in science, engineering, physics and biology which can be described by differential equation (DEs). Approximation analytical methods such as Variational Iteration Method (VIM), Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM) have been used to solve nonlinear DE involve BVP (He, 1999). In 1998 the HPM first proposed by Ji Huan He solves differential equations (He,2003; He, 2000). HPM applied for nonlinear Ordinary differential equation (He, 2000), Partial differential equation (He, 2004A), bifurcation (He, 2004A), Integration differential equations (He, 2004B), with Initial value problems (IVP) (He, 2005). DDEs with BVP has been investigated by the Laplace Adomian decomposition method (LADM) with the second order of BVP (Kanth et al.,2009). Aslamnoor and Mohyud presented DEs solution with BVP using HPM (Bellen, 2003). In this paper, some examples of DDEs have been solved and compared with the exact solution to illustrate the capability and reliability of this method. We have found that HPM is a powerful mathematical tool used for various DDEs. HPM, which provides an approximate solution to the ordinary and partial differential equations also DDEs with BVP.

II. DELAY DIFFERENTIAL EQUATIONS

In this study, DDEs were commonly involved in the mathematical formulation of physical and engineering phenomena. DDEs in mathematics are a type of differential equation in which the derivative of the unknown function at a particular time is given regarding the values of the equation at previous times. Also, DDEs except that they involve past values of variables and/or derivatives (Xiuying, 2008; Cveticanin,2006; Kuang, 1993; Nayfeh,1985; Mohyud-Din & Aslam, 2008; Agarwal,1986). Here, we examine the general form of DDEs with BVP by considering as the following:

$$y''(x) = g(x)\, y'(x) + h(x)\, y(qx) + f(x), \quad x > 0, \ x \in R^n \tag{1}$$

with Boundary condition's

$$y(0) = a, \ y(1) = b. \tag{2}$$

*Corresponding author's e-mail: hamood102@yahoo.com

where $g(x), h(x)$ and $f(x)$ are an analytical function $0 < q < 1$ and a, b are constant.

III. PADÉ APPROXIMATION

In this thesis, the Padé approximation is used to provide a better estimate of the function than truncating. Padé approximation is used extensively in computer calculations. The polynomial in the denominator allows an approximation of functions that have rational singularities that can be thought of as a generalization of a Taylor polynomial (Chisholm,1973). Because these functions only use the elementary arithmetic operations, they are easy to evaluate numerically. A Padé approximate is that rational function whose power series expansion agrees with a prescribed power series to the highest possible order.

Definition.1 The Padé approximation is a rational of two polynomials $P_L(r)$ and $Q_M(r)$ in degree $N = L + M + 1$. It is denoted by [L/M]u(r) for an analytical function with Taylor or Maclaurin's expansion, constructed from the coefficients of the series expansion of a function (George Jr, 1975). Let assume that $u(r)$ an analytical function represents as a Taylor expansion $u(r) = \sum_{i=1}^{\infty} u_i r^i$ it has a truncated to m terms $P_m(r)$

$$u(r) - b_L(r) = O(r^{L+1}). (3)$$

Then Padé approximant $R_{L/M}(r)$ can be written as follows

$$R_{L/M}(r) = \left[\frac{L}{M}\right] u(r) = \frac{P_L(r)}{Q_M(r)} - o\left(r^{L+M+1}\right) (4)$$

which we denote by powerful series can be transformed into Padé series is defined as we ignore $O\left(r^{L+1}\right)$

$$\left[\frac{L}{M}\right] u(r) = \frac{P_L(r)}{Q_M(r)} = \frac{\sum_{i=0}^{L} p_i r^i}{\sum_{j=0}^{M} q_j r^j} (5)$$

We set $q_0 = 1$ it is becoming,

$$a_0 + a_1 r + a_2 r^2 + \dots = \frac{p_0 + p_1 r \dots + p_M r^L}{1 + q_1 r \dots + q_L r^M}. (6)$$

Multiply both sides of Eq.(6)by the to the denominator of the coefficients p_L and q_M where

$$
\begin{aligned}
P_0 &= a_l, \\
P_l &= q_1 a_0 + a_l, \\
&\vdots \\
P_n &= q_m a_{n-m} + \dots + q_1 a_{n-1} + a_m,
\end{aligned}
\quad (7)
$$

In this paper, Padé approximation is obtained by the use of Mathematica software of various orders. Solving the linear Eq. (6), we have to calculate approximate such as [2/2], [3/3], and [4/4], ...(Baker et al., 1996).

IV. HOMOTOPY PERTURBATION METHOD

The aim of this section is to obtain the approximate solution using the HPM for solving the second order BVP involving DDEs and compare the results obtained with the exact solution. To illustrate HPM, we consider the following general type of DE:

$$D(y) - g(r) = 0, r \in \Omega, (8)$$

with the boundary conditions

$$B(y, \frac{\partial y}{\partial n}) = 0, \quad r \in \Gamma, (9)$$

where D is a general differential operator, it can be divided into linear parts L and N nonlinear term of Eq. (8) B a boundary operator, $g(r)$ and a known analytical function and Γ is the boundary of the domain Ω. Can be rewritten as:

$$L(y) + N(y) - g(r) = 0, (10)$$

By HPM, construct a Homostopy $v(r,q): \Omega \times [0,1] \to R$, which satisfies:

$$H(v,q) = (1-q)[L(v) - L(v_0)] + q[A(v) - g(r)] = 0, \quad (11)$$

or

$$H(v,q) = L(v) - L(v_0) + qL(v_0) + q[N(v) - g(r)] = 0 \quad (12)$$

where $q \in [0,1]$ is an embedding parameter, v_0 is an initial condition, which satisfies the boundary conditions. From

Eq. (11) or Eq. (12) we will have

$$H(v,0) = L(v) - L(v_0) = 0,$$
$$H(v,1) = D(v) - g(r) = 0. \tag{13}$$

We assume that the solution of Eq. (112) can be written as a power series in q

$$v(x) = v_0(x) + qv_1(x) + q^2 v_2(x) + q^3 v_3(x) + \cdots, \tag{14}$$

Moreover, setting $q = 1$ obtained the approximate solutions follows:

$$y = \lim_{q \to 1} v = \lim_{q \to 1}(v_0 + qv_1 + q^2 v_2 + q^3 v_3 + \cdots)$$
$$= \sum_{j=0}^{\infty} v_j, \tag{15}$$

The series (2) is convergent for most cases and the rate of convergence depends on A(v) (He, 2006). Using the Pade approximation for Eq. (15) by definition section (III)

$$u_n(x) = \left[\frac{L}{M}\right] \tag{16}$$

V. NUMERICAL EXAMPLES

In this section, we will apply the HPM on two examples involving second order DDEs with the second order BVP that have been solved by ADM and VIM. We present two examples to show the efficiency and high accuracy of the present method, camper errors the solution obtained by HPM and exact solution.

Example .1 Consider the linear DDEs with BVP as the following

$$u''(x) = 0.5u(x) + e^{-0.5x}(u_0(x)/2) - 2e^{-x}, \tag{17}$$

the boundary conditions

$$u(0) = 0, u(1) = e^{-1}. \tag{18}$$

The analytical solution

$$u(x) = x e^{-x}, \tag{19}$$

According to the HPM and using Eq. (11-14), we have the solutions we construct a homotopy as follows:

$$(1-q)\left(\frac{\partial^2 u}{\partial x^2} - 2e^{-x}\right) + q\left(\frac{\partial^2 u}{\partial x^2} - 0.5u + e^{-0.5x}(u_n(.5x))\right) = 0, \tag{20}$$

Assume the solution of Eq. (17) has the form

$$u(x) = u_0(x) + qu_1(x) + q^2 u_2(x) + q^3 u_3(x) + \cdots$$
$$= \sum_{j=0}^{\infty} q^j u_j(x), \tag{21}$$

where $u_j(x)$ are functions can be determined.

$$q^0 : \frac{\partial^2 u_0}{\partial x^2} = \frac{\partial u_0}{\partial x}, \quad u_0(0) = 0, \tag{22}$$

$$q^1 : \frac{\partial^2 u_1}{\partial x^2} = 0.5u - e^{-0.5x} u_n(.5x),$$
$$u_1(0) = 0, \tag{23}$$

$$q^4 : \frac{\partial^2 u_4}{\partial x^2} = 0.5u - e^{-0.5x} u_n(.5x),$$
$$u_4(0) = 0, \tag{24}$$

where $u_{k+1}(0) = 0, k \geq 1$ considering $u_0(x) = u(0) = f(x)$, we have

$$u(0) = u_0(x) = 0, \tag{25}$$

Using Pade approximation for Eq. (22-24) by definition

$$u_n(x) = \left[\frac{L}{M}\right] = x - x^2 + \frac{x^3}{2} + \frac{x^4}{6} \tag{26}$$

Table 1. Comparisons of HPM with an exact solution by 4-termsfor Example. (1)

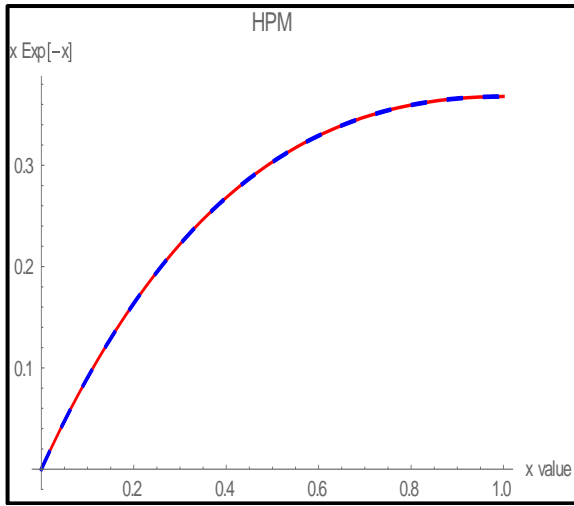| x	Exact solution	Approximate [HPM]	Absolute Error
0	0	$-3.410605131648481 \times 10^{-13}$	$3.410605131648481 \times 10^{-13}$
0.2	0.26812801841425576	0.2681280184139023	$3.534394998894186 \times 10^{-13}$
0.4	0.35946317129377725	0.3594631711375307	$1.562465712368066 \times 10^{-10}$
0.6	0.3614330542946425	0.36143304174098034	$1.255366216534881 \times 10^{-8}$
0.8	0.3230344287914486	0.32303415205801045	$2.767334381648112 \times 10^{-7}$
1	0.2706705664732254	0.27066756119533864	$3.0052778867695196 \times 10^{-6}$

Figure 1. Absolute errors for HPMs with Exact solution four terms of example (1)

The comparison between the approximate solutions obtained by HPM and the exact solution was given in table (1) and figure (1) it shows the graphical representation the approximate solutions is very close to the exact solution.

Example.2 Consider the linear DDEs with BVP the following

$$\frac{d^2u}{dx^2} = 1 + 2(1 + \frac{1}{8}x^2)\, cos\,(\frac{1}{2}x) - 2cos\,(\frac{1}{2}x)u(\frac{1}{2}x)$$

$$x \in [0, \frac{\pi}{4}] \quad (27)$$

subject to the boundary conditions

$$u(0) = 1,\, u(\frac{\pi}{4}) = 1 + \frac{\sqrt{2}}{2} + \frac{\pi^2}{32}. \quad (28)$$

The analytical solution

$$u(x) = 1 + \frac{1}{2}x^2 + sin(x) \quad (29)$$

According to the HPM, the construction of homotopy is as follows

$$(1-q)\left(\frac{\partial^2 u}{\partial x^2} - 1\right) +$$

$$q\left(\frac{\partial^2 u}{\partial x^2} - 1 - 2(1 + \frac{x^2}{8})\, cos\,(\frac{x}{2}) + 2cos\,(\frac{x}{2})u(\frac{x}{2})\right) = 0, \quad (30)$$

Assume the solution of Eq. (19) has the form

$$u(x) = u_0(x) + qu_1(x) + q^2u_2(x) + q^3u_3(x) + \cdots$$

$$= \sum_{j=0}^{\infty} q^j u_j(x), \quad (31)$$

where $u_j(x)$ are functions yet to be determined., we have

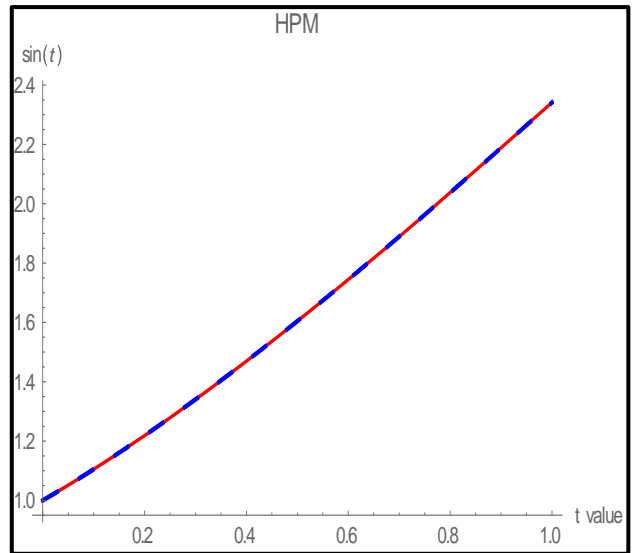$$u_n(x) = \left[\frac{L}{M}\right] = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^5}{120} \quad (32)$$

Figure 2. The errors between results obtained using HPM and the exact solution with four terms.

Table 2. Comparisons of HPM with an exact solution by 4-terms for Example. (2)

x	Analytical solution	Approximate [HPM]	Absolute Error
0	0	$-3.410605131648481 \times 10^{-13}$	$3.410605131648481 \times 10^{-13}$
0.2	0.26812801841425576	0.2681280184139023	$3.534394998894186 \times 10^{-13}$
0.4	0.35946317129377725	0.3594631711375307	$1.562465712368066 \times 10^{-10}$
0.6	0.3614330542946425	0.36143304174098034	$1.255366216534881 \times 10^{-8}$
0.8	0.3230344287914486	0.32303415205801045	$2.767334381648112 \times 10^{-7}$
1	0.2706705664732254	0.27066756119533864	$3.0052778867695196 \times 10^{-6}$

The previous results show that the HPM is efficient and simple through a few terms of the series that used to get the approximate solutions.

VI.   CONVERGENCE OF HPM

We introduce the study of the problem of convergence of the HPM.

Theorem.1 Let B a Banach space.

a- $\sum_{i=0}^{\infty} v_i$ obtained by (18), convergence to B, if $\sum_{i=0}^{\infty} v_i$, satisfies in

b- $S = L^{-1}N(s + v_0) + u_0 + L^{-1}(f(t))$

Proof. (a) we have

$$\forall v, \bar{v} \in X \ ; \ \|S_{n+1}(t) - S_n(t)\| \le r\|v - \bar{v}\|, 0 < r < 1.$$

$$\|S_{n+1} - S_n\| = \|u_{n+1}(t)\| \le r\|u_n(t)\| \le r^2\|u_{n-1}(t)\| \le \cdots$$
$$\le r^{n+1}\|u_0(t)\| \tag{33}$$

For any, $n.m \in N$, $n \ge m$ we drive

$$\|\varphi_{n+1} - \varphi_n\| = \|(\varphi_{n+1} - \varphi_n) + (\varphi_{n+1} - \varphi_n) + \cdots$$
$$+ (\varphi_{n+1} - \varphi_n)\|$$
$$\le \|\varphi_{n+1} - \varphi_n\| + \|\varphi_{n+1} - \varphi_n\|$$

and

$$\|\varphi_{n+1} - \varphi_n\| \le \|(\varphi_n - \varphi_{n-1})\| + \|(\varphi_{n-1} - \varphi_{n-2})\| + \cdots$$
$$+ \|(\varphi_{m+1} - \varphi_m)\|$$
$$\le r\|\varphi_n\| + r^2\|\varphi_{n-1}\| + \cdots + r^{n+1}\|\varphi_0\| \tag{34}$$
$$\le (r^{m+1} + r^{m+2} + \cdots)\|\varphi_0\|$$
$$= \frac{1 - r^{n-m}}{1 - r}\|\varphi_0\|$$

So

$$\lim_{n,m \to \infty} \|(V_n - u)\| = 0, \tag{35}$$

Therefore, $\{S_n\}_{n=0}^{\infty}$ a Cauchy sequence in Banash space and it is convergent.

VII.   CONVERGENCE OF HPM

The HPM convergence to the exact result is studied in this section. The series solution
$$u(x) = \lim_{p \to 1} v(x) = v + v_0 + v_1 + v_2 + \cdots$$
obtained by HPM converge to the exact solution. The homotopy standard for Eq. (2) has a scheme

$$u_0 = u_0$$
$$u_1 = L^{-1}(f(x) - u_0)$$
$$u_2 = -L^{-1}(N_1(x)) \tag{36}$$
$$\vdots$$
$$u_{n+1} = -L^{-1}(N_n(x))$$

Theorem.1 (Ayati & Biazar,, 2015)Let A be a Banach space. The series solution $\sum_{m=1}^{+\infty} u_m(t)$ converges to the exact solution $s \in A$ if $\exists (0 \le \gamma < 1)$ such that $\|u_{n+1}\| \le \gamma\|u_n\|, \ \forall n \in \Box \ \bigcup\{0\}$.

Theorem.2(Ayati and Biazar, 2015) for every $n \in \Box \ \bigcup\{0\}$, the parameters define as

$$\gamma_n = \begin{cases} \dfrac{\|u_{n+1}\|}{\|u_n\|}, & \|u_n\| \ne 0 \\ 0, & \|u_n\| = 0 \end{cases} \tag{37}$$

In Theorem .2 the DDE series solution

$\sum_{m=0}^{\infty} u_m(t)$ converges to the exact solution, when $u_n(t)$.

For the $0 \le \gamma_n \le 1, \ \forall n \in \Box \ \bigcup\{0\}$.

VIII.   RATIO TEST CONVERGENCE

The ratio test can be used to study the convergence of the approximate LTHPM. Let the sequence be as according to the above Theorem 6.2. We will study the convergence of

HPM and HPM by using the ratio test for examining the conditions $0 \le \gamma_k \le 1, \ k = 0,1,2,3,\cdots$ for some examples of DDEs.

IX. APPLICATION OF RATIO TEST OF CONVERGENCE

Example 1 We check the convergence of the HPM for example 1

$$\gamma_0 = \left\|\frac{u_2}{u_1}\right\| = 0.3982570635154341 < 1$$

$$\gamma_1 = \left\|\frac{u_3}{u_2}\right\| = 0.21192630386316658 < 1$$

$$\gamma_2 = \left\|\frac{u_4}{u_3}\right\| = 0.13996588393175585 < 1$$

$$\vdots$$

Example.2 We check the convergence of the HPM for example 2

$$\gamma_0 = \left\|\frac{u_1}{u_0}\right\| = 0.03622530276953514 < 1$$

$$\gamma_1 = \left\|\frac{u_2}{u_1}\right\| = 0.013694821336785339 < 1$$

$$\gamma_2 = \left\|\frac{u_3}{u_2}\right\| = 0.0074362023444989765 < 1$$

$$\vdots$$

where γ_1, γ_2 and γ_3 are less than one. So, the approximate HPM of DDEs are convergent.

X. SUMMARY

In this paper, HPM is a powerful approximate method was introduced to obtain an analytical approach for DDEs with BVPs. The scheme based HPM has been formulated to get the solution of DDEs directly without reducing into first order systems. The HPM has been successfully used to study some examples of DDEs. The HPM avoids Adomian polynomials which can be difficult in some cases. The results show that the proposed method is a more workable mathematical tool for finding the exact and approximate solutions of nonlinear equations. The study of the convergence to the exact solution for DDEs with BVP briefly addressed. The HPM is very useful and convenient method.

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