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Some Coefficient Problems on Bi-univalent Functions

Andy Liew Pik Hern* and Aini Janteng

Faculty of Science and Natural Resources, Universiti Malaysia Sabah, 88400 Kota Kinabalu, Sabah, Malaysia.

In the open unit disk $D=\{z\epsilon\mathbb{C}: |z|<1\}$, we denote A be the class of all analytic functions $f(z)$.We consider conditions $f(0)=0$ and $f'(0)=1$ so called normalized condition and denote Σ as the class of bi-univalent functions defined in D. If both $f(z)$ and $f^{-1}(f(z))$ are univalent in D, we say that a function $f\in A$ to be bi-univalent in D. In this paper, some subclasses of bi-univalent functions are introduced. Coefficient estimates on $|a_2|$ and $|a_3|$ are determined. In addition, the upper bounds of the Fekete-Szegö functional are also obtained.

Keywords: analytic functions, bi-univalent functions, coefficient estimates, Fekete-Szegö functional

I. INTRODUCTION

In this paper, the class of functions $f(z)$ which are analytic in the open unit disk $D=\{z\epsilon\mathbb{C}: |z|<1\}$ in the form of

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \quad (z\in D) \tag{1}$$

is denoted by A. We denote $S\subset A$ which are univalent in D as well. According (Duren, 1983), the Koebe one-quarter theorem showed the image of D under every univalent functions f in S contains a disk of radius $1/4$. So, an inverse for every univalent function f can be defined as

$$f^{-1}(f(z))=z$$

and

$$f(f^{-1}(w))=w, \quad (|w|<r_o(f), r_o(f)\geq 1/4).$$

Thus, for a function $f\in A$, if both f and f^{-1} are univalent in D, this function is bi-univalent function. The notation of bi-univalent is Σ.

Motivated by the previous works, for example (Shanmugam & Sivasubramanian, 2005), (Janteng et al., 2006), (Aouf et al., 2013), (Zaprawa, 2014) and (Altinkaya & Yalcin, 2017),we consider the following subclasses of Σ.

Definition 1. A function $f(z)\in\Sigma$ in (1) is said to be in the class $A_{\Sigma}(\alpha,\lambda)$ with $0<\alpha\leq 1$ and $\lambda\geq 0$ if: $f\in\Sigma$ and

$$\left|\arg\left(\frac{\lambda z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2}, \quad z\in D \tag{2}$$

and

$$\left|\arg\left(\frac{\lambda w^2 g''(w)}{g(w)}+\frac{wg'(w)}{g(w)}\right)\right|<\frac{\alpha\pi}{2}, \quad w\in D \tag{3}$$

where

$$g(w)=w-a_2w^2+\left(2a_2{}^2-a_3\right)w^3 -\left(5a_2{}^3-5a_2a_3+a_4\right)w^4+\ldots \tag{4}$$

Definition 2. A function $f(z)\in\Sigma$ in (1) is said to be in class $A_{\Sigma}(\beta,\lambda)$ with $0\leq\beta<1$ and $\lambda\geq 0$ if:

$$\operatorname{Re}\left(\frac{\lambda z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right)>\beta \tag{5}$$

and

$$\operatorname{Re}\left(\frac{\lambda w^2 g''(w)}{g(w)}+\frac{wg'(w)}{g(w)}\right)>\beta. \tag{6}$$

Definition 3. A function $f(z)\in\Sigma$ in (1) is said to be in the class $B_{\Sigma}(\alpha,\lambda)$ with $0<\alpha\leq 1$ and $\lambda\geq 0$ if: $f\in\Sigma$ and

$$\left|\arg\left(\frac{\left(\lambda z^2 f''(z)+zf'(z)\right)'}{f'(z)}\right)\right|<\frac{\alpha\pi}{2}, \quad z\in D \tag{7}$$

*Corresponding author's e-mail: andyliew1992@yahoo.com

and

$$\left|\arg\left(\frac{\left(\lambda w^2 g''(w)+wg'(w)\right)'}{g'(w)}\right)\right|<\frac{\alpha\pi}{2},\quad w\in D. \quad (8)$$

Definition 4. A function $f(z)\in\sum$ in (1) is said to be in class $B_{\Sigma}(\beta,\lambda)$ with $0\le\beta<1$ and $\lambda\ge 0$ if:

$$\mathrm{Re}\left(\frac{\left(\lambda z^2 f''(z)+zf'(z)\right)'}{f'(z)}\right)>\beta \quad (9)$$

and

$$\mathrm{Re}\left(\frac{\left(\lambda w^2 g''(w)+wg'(w)\right)'}{g'(w)}\right)>\beta. \quad (10)$$

This paper obtained the upper bound for coefficients $|a_2|$ and $|a_3|$ and Fekete-Szegö functional for f in the subclasses of $\sum$.

II. METHODS

The following lemmas are required to get the main results.

Lemma 1. (Duren, 1983) If $p\in P$ then $|p_k|\le 2$ for each k, where P is the family of all functions p analytic in D, $\mathrm{Re}(p(z))>0$, $p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots$ for $z\in D$.

Lemma 2. (Zaprawa, 2014) Let $k,l\in\mathbb{R}$ and $z_1,z_2\in\mathbb{C}$. If $|z_1|<R$ and $|z_2|<R$ then

$$\left|(k+l)z_1+(k-l)z_2\right|\le\begin{cases}2|k|R & \text{for } |k|\ge|l|\\ 2|l|R & \text{for } |k|\le|l|\end{cases}.$$

III. RESULTS

The main result for $f\in A_{\Sigma}(\alpha,\lambda)$ is stated as follows.

Theorem 1. Let f in (1) be in the class $A_{\Sigma}(\alpha,\lambda)$ where $0<\alpha\le 1$ and $\lambda\ge 0$. Then

$$|a_2|\le\frac{2\alpha}{\sqrt{4\lambda\left[\lambda(1-\alpha)+\alpha+1\right]+\alpha+1}} \quad (11)$$

and

$$|a_3|\le\frac{4\alpha^2}{(2\lambda+1)^2}+\frac{\alpha}{3\lambda+1}. \quad (12)$$

Proof. It follows from (2) and (3) that

$$\frac{\left(\lambda z^2 f''(z)+zf'(z)\right)'}{f'(z)}=\left[p(z)\right]^{\alpha} \quad (13)$$

and

$$\frac{\left(\lambda w^2 g''(w)+wg'(w)\right)'}{g'(w)}=\left[q(w)\right]^{\alpha} \quad (14)$$

where $p(z)$ and $q(w)$ in P have the forms $p(z)=1+p_1z+p_2z^2+\ldots$ and $q(w)=1+q_1w+q_2w^2+q_3w^3+\ldots$ respectively.

From (1), we may get

$$\frac{\lambda z^2 f''(z)+zf'(z)}{f(z)}=1+a_2(2\lambda+1)z+\left[2a_3(3\lambda+1)-a_2^2(2\lambda+1)\right]z^2+\ldots \quad (15)$$

and from (4) we can get

$$\frac{\lambda w^2 g''(w)+wg'(w)}{g(w)}=1-a_2(2\lambda+1)w+\begin{bmatrix}2a_3\left(2a_2^2-a_3\right)(3\lambda+1)\\ -a_2^2(2\lambda+1)\end{bmatrix}w^2+\ldots. \quad (16)$$

Hence, equations (17) and (18) give

$$1+a_2(2\lambda+1)z+\left[2a_3(3\lambda+1)-a_2^2(2\lambda+1)\right]z^2+\ldots=1+\alpha p_1z+\left[\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2\right]z^2+\ldots \quad (17)$$

and

$$1-a_2(2\lambda+1)w+\begin{bmatrix}2a_3\left(2a_2^2-a_3\right)(3\lambda+1)\\ -a_2^2(2\lambda+1)\end{bmatrix}w^2+\ldots=1+\alpha q_1w+\left[\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2\right]w^2+\ldots \quad (18)$$

Next, by suitably comparing coefficients of z and z^2 in (17) and comparing coefficients of w and w^2 in (18), we get

$$a_2(2\lambda+1)=\alpha p_1, \quad (19)$$

$$2a_3(3\lambda+1)-a_2^2(2\lambda+1)=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2, \quad (20)$$

$$-a_2(2\lambda+1)=\alpha q_1, \quad (21)$$

$$2\left(2a_2^2-a_3\right)(3\lambda+1)-a_2^2(2\lambda+1)=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \quad (22)$$

By dividing (19) and (21), we get

$$p_1=-q_1.$$

Next, by adding the square of equations (19) and (21), we may get

$$2a_2^2(2\lambda+1)^2=\alpha^2\left(p_1^2+q_1^2\right). \quad (23)$$

Now, by adding of equations (20) and (22), we find that

$$2a_3(3\lambda+1)-a_2^2(2\lambda+1)+4\alpha_2^2(3\lambda+1)$$
$$-2a_3(3\lambda+1)-a_2^2(2\lambda+1) \quad (24)$$
$$=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)$$

Then, from (23), by replacing $p_1^2+q_1^2$ into equation (24), we obtain

$$a_2^2=\frac{\alpha^2(p_2+q_2)}{4\lambda[\lambda(1-\alpha)+\alpha+1]+\alpha+1} \quad (25)$$

By applying triangle inequality and Lemma 1.1 for the coefficients p_2 and q_2 into equation (25), we finally get:

$$|a_2|\le\frac{2\alpha}{\sqrt{4\lambda[\lambda(1-\alpha)+\alpha+1]+\alpha+1}}.$$

This gives the bound on $|a_2|$ in (11). Next, to find $|a_3|$, by subtracting (22) from (20), we get

$$4a_3(3\lambda+1)-4a_2^2(3\lambda+1)$$
$$=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2)$$

Since $p_1=-q_1$ then $p_1^2=q_1^2$. Next, we have

$$a_3=a_2^2+\frac{\alpha(p_2-q_2)}{4(3\lambda+1)} \quad (26)$$

From (23), we can get $a_2^2=\dfrac{\alpha^2(p_1^2+q_1^2)}{2(2\lambda+1)^2}$.

Thus, by substituting $a_2^2=\dfrac{\alpha^2(p_1^2+q_1^2)}{2(2\lambda+1)^2}$ into equation (26), we obtain

$$a_3=\frac{\alpha^2(p_1^2+q_1^2)}{2(2\lambda+1)^2}+\frac{\alpha(p_2-q_2)}{4(3\lambda+1)} \quad (27)$$

Once again, applying triangle inequality and Lemma 1.1 for the coefficients p_1, p_2, q_1 and q_2 into equation (27), we get:

$$|a_3|\le\frac{4\alpha^2}{(2\lambda+1)^2}+\frac{\alpha}{3\lambda+1}.$$

Theorem 1 is completely proven.

Taking $\lambda=0$ in Theorem 1, we obtain the following corollary.

Corollary 1 (Murugusundaramoorthy & Magesh, 2009)Let $f(z)$in (1) be in the class $SS_\Sigma^*(\alpha)$ and $0<\alpha\le1$. Then

$$|a_2|\le\frac{2\alpha}{\sqrt{\alpha+1}},$$

and

$$|a_3|\le4\alpha^2+\alpha.$$

Next, we obtained Theorem 2, Theorem 3 and Theorem4.

Theorem 2 Let f in (1) be in the class $A_\Sigma(\beta,\lambda)$ where $0\le\beta<1$ and $\lambda\ge0$. Then

$$|a_2|\le\sqrt{\frac{2(1-\beta)}{4\lambda+1}} \quad (28)$$

and

$$|a_3|\le\frac{4(1-\beta)^2}{(2\lambda+1)^2}+\frac{1-\beta}{3\lambda+1}. \quad (29)$$

Taking $\lambda=0$ in Theorem 2, we obtain the following corollary.

Corollary 2 (Murugusundaramoorthy & Magesh, 2009)Let $f(z)$in (1) be in the class $S_\Sigma^*(\beta)$ and $0\le\beta<1$. Then

$$|a_2|\le\sqrt{2(1-\beta)}$$

and

$$|a_3|\le4(1-\beta)^2+1-\beta.$$

Theorem 3 Let f in (1) be in the class $B_\Sigma(\alpha,\lambda)$ where $0<\alpha\le1$ and $\lambda\ge0$. Then

$$|a_2|\le\frac{\alpha}{\sqrt{\alpha(5\lambda+1)+(1-\alpha)(2\lambda+1)^2}} \quad (30)$$

and

$$|a_3|\le\frac{\alpha^2}{(2\lambda+1)^2}+\frac{\alpha}{3(3\lambda+1)} \quad (31)$$

Taking $\lambda=0$ in Theorem 3, we obtain the following corollary.

Corollary 3 (Zaprawa, 2014) Let $f(z)$in (1) be in the class $B_\Sigma(\alpha)$ and $0<\alpha\le1$. Then

$$|a_2|\le\alpha$$

and

$$|a_3|\le\alpha^2+\frac{\alpha}{3}.$$

Theorem 4 Let f in (1) be in the class $B_\Sigma(\beta,\lambda)$ where $0<\alpha\le1$ and $\lambda\ge0$. Then

$$|a_2|\le\sqrt{\frac{1-\beta}{5\lambda+1}}, \quad (32)$$

and

$$|a_3|\le\frac{2(1-\beta)^2}{(2\lambda+1)^2}+\frac{1-\beta}{3(3\lambda+1)}. \quad (33)$$

Taking $\lambda=0$ in Theorem 4, we obtain the following corollary.

Corollary 4 Let $f(z)$in (1) be in the class $B_\Sigma(\beta)$ and $0\le\beta<1$. Then

$$|a_2|\le\sqrt{1-\beta}$$

and

$$|a_3| \le 2(1-\beta)^2 + \frac{1}{3}(1-\beta).$$

Our second main results for the classes $A_\Sigma(\alpha,\lambda)$, $A_\Sigma(\beta,\lambda)$, $B_\Sigma(\alpha,\lambda)$ and $B_\Sigma(\beta,\lambda)$ are given by Theorem 5, Theorem 6, Theorem 7 and Theorem 8.

Theorem 5 If $f \in A_\Sigma(\alpha,\lambda)$, $0<\alpha\le 1$, $\lambda \ge 0$ and $\mu \in \mathbb{R}$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{4\alpha^2}{2\alpha(4\lambda+1)-(\alpha-1)(2\lambda+1)^2}|1-\mu| & \text{for } 4\alpha(3\lambda+1)|(1-\mu)| \ge 2\alpha(4\lambda+1)-(\alpha-1)(2\lambda+1)^2, \\ \dfrac{\alpha}{3\lambda+1} & \text{for } 4\alpha(3\lambda+1)|(1-\mu)| \le 2\alpha(4\lambda+1)-(\alpha-1)(2\lambda+1)^2. \end{cases} \quad (34)$$

Proof. Let f given by (1) be in $A_\Sigma(\alpha,\lambda)$, $0<\alpha\le 1$, $\lambda\ge 0$ and $\mu\in\mathbb{R}$.
By adding of equations (20) and (22), we find that

$$2a_2^2(4\lambda+1) = \alpha(p_2+q_2) + \frac{1}{2}\alpha(\alpha-1)(p_1^2+q_1^2) \quad (35)$$

Next, by subtracting of equations (22) from (20), we get

$$a_3 = a_2^2 + \frac{\alpha(p_2-q_2)}{4(3\lambda+1)} \quad (36)$$

From equation (23), we have

$$p_1^2 + q_1^2 = \frac{2a_2^2(2\lambda+1)^2}{\alpha^2} \quad (37)$$

By substituting equation (37) into equation (35), yields

$$a_2^2 = \frac{\alpha^2}{2\alpha(4\lambda+1)-(\alpha-1)(2\lambda+1)^2}(p_2+q_2) \quad (38)$$

By substituting equation (38) into equation (36),

$$a_3 = \left[\frac{\alpha^2}{2\alpha(4\lambda+1)-(\alpha-1)(2\lambda+1)^2}\right](p_2+q_2) + \frac{\alpha}{4(3\lambda+1)}(p_2-q_2) \quad (39)$$

Thus, from equations (38) and (39), we obtain

$$\begin{aligned} a_3-\mu a_2^2 &= \left[\frac{\alpha^2}{2\alpha(4\lambda+1)-(\alpha-1)(2\lambda+1)^2}\right](p_2+q_2) + \frac{\alpha}{4(3\lambda+1)}(p_2-q_2) \\ &\quad -\mu\left[\frac{\alpha^2}{2\alpha(4\lambda+1)-(\alpha-1)(2\lambda+1)^2}(p_2+q_2)\right] \\ &= h(\alpha,\lambda)(1-\mu)(p_2+q_2) + \frac{\alpha}{4(3\lambda+1)}(p_2-q_2) \end{aligned}$$

where $h(\alpha,\lambda) = \dfrac{\alpha^2}{2\alpha(4\lambda+1)-(\alpha-1)(2\lambda+1)^2}$ is nonnegative.
Hence,

$$\begin{aligned} a_3-\mu a_2^2 &= h(\alpha,\lambda)(1-\mu)p_2 + h(\alpha,\lambda)(1-\mu)q_2 + \frac{\alpha}{4(3\lambda+1)}p_2 - \frac{\alpha}{4(3\lambda+1)}q_2 \\ &= \left[h(\alpha,\lambda)(1-\mu) + \frac{\alpha}{4(3\lambda+1)}\right]p_2 + \left[h(\alpha,\lambda)(1-\mu) - \frac{\alpha}{4(3\lambda+1)}\right]q_2 \end{aligned}$$

From Lemma 1 and Lemma 2, we obtain

$$\begin{aligned} |a_3-\mu a_2^2| &= \left| \left[h(\alpha,\lambda)(1-\mu) + \frac{\alpha}{4(3\lambda+1)}\right]p_2 + \left[h(\alpha,\lambda)(1-\mu) - \frac{\alpha}{4(3\lambda+1)}\right]q_2 \right| \\ &\le \begin{cases} 2|h(\alpha,\lambda)(1-\mu)|(2) & \text{for } |h(\alpha,\lambda)(1-\mu)| \ge \dfrac{\alpha}{4(3\lambda+1)} \\ 2\left|\dfrac{\alpha}{4(3\lambda+1)}\right|(2) & \text{for } |h(\alpha,\lambda)(1-\mu)| \le \dfrac{\alpha}{4(3\lambda+1)} \end{cases} \\ &= \begin{cases} 4h(\alpha,\lambda)|1-\mu| & \text{for } h(\alpha,\lambda)|(1-\mu)| \ge \dfrac{\alpha}{4(3\lambda+1)} \\ \dfrac{\alpha}{3\lambda+1} & \text{for } h(\alpha,\lambda)|(1-\mu)| \le \dfrac{\alpha}{4(3\lambda+1)} \end{cases} \end{aligned}$$

Theorem 5 is completely proven.

Taking $\mu = 0$ and $\lambda = 0$ in Theorem 5, we obtain the following corollary.

Corollary 5 If $f(z)$ in equation (1) be in the class $SS_\Sigma^*(\alpha)$, $0<\alpha\le 1$, then

$$|a_3| \le \begin{cases} \alpha & \text{for } 0<\alpha\le \dfrac{1}{3} \\ \dfrac{4\alpha^2}{\alpha+1} & \text{for } \dfrac{1}{3}\le\alpha\le 1 \end{cases}.$$

The result in Corollary 5 is similar to Corollary 10 in (Zaprawa, 2014) if $\lambda = 0$.

Putting $\alpha = 1$ in Corollary 5, we will get the following corollary.

Corollary 6 (Zaprawa, 2014) If $f(z)$ in equation (1) be in the class $SS_{\Sigma}^{*}(1)$, then

$$|a_3| \le 2.$$

Next, we obtained Theorem 6, Theorem 7 and Theorem 8 as follows.

Theorem 6 If $f \in A_{\Sigma}(\beta,\lambda)$, $0 \le \beta < 1$, $\lambda \ge 0$ and$\mu\epsilon\mathbb{R}$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{2(1-\beta)}{(4\lambda+1)}|1-\mu| & \text{for } 2(3\lambda+1)|1-\mu| \ge 4\lambda+1, \\ \dfrac{1-\beta}{3\lambda+1} & \text{for } 2(3\lambda+1)|1-\mu| \le 4\lambda+1. \end{cases} \tag{40}$$

Taking $\mu = 0$ and $\lambda = 0$ in Theorem 6, we obtain the following corollary.

Corollary 7 If $f(z)$ in equation (1) be in the class $S_{\Sigma}^{*}(\beta)$, $0 \le \beta < 1$, then

$$|a_3| \le 2(1-\beta).$$

The result in Corollary 7 is similar to Corollary 11 in (Zaprawa, 2014) if $\lambda = 0$.

Theorem 7 If $f \in B_{\Sigma}(\alpha,\lambda)$, $0 < \alpha \le 1$, $\lambda \ge 0$ and$\mu\epsilon\mathbb{R}$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{\alpha^2}{\alpha(5\lambda+1)+(1-\alpha)(2\lambda+1)^2}|1-\mu| & \text{for } 3\alpha(3\lambda+1)|(1-\mu)| \\ & \quad \ge \alpha(5\lambda+1)+(1-\alpha)(2\lambda+1)^2, \\ \dfrac{\alpha}{3(3\lambda+1)} & \text{for } 3\alpha(3\lambda+1)|(1-\mu)| \le \alpha(5\lambda+1)+(1-\alpha)(2\lambda+1)^2. \end{cases} \tag{41}$$

Taking $\mu = 0$ and $\lambda = 0$ in Theorem 7, we obtain the following corollary.

Corollary 8 If $f(z)$ in equation (1) be in the class $B_{\Sigma}(\alpha)$, $0 < \alpha \le 1$, then

$$|a_3| \le \begin{cases} \dfrac{1}{3}\alpha & \text{for } 0 < \alpha \le \dfrac{1}{3} \\ \alpha^2 & \text{for } \dfrac{1}{3} \le \alpha \le 1 \end{cases}.$$

The result in Corollary 8 is similar to Corollary 17 in (Zaprawa, 2014) if $\lambda = 0$.

Putting $\alpha = 1$ in Corollary 8, we will get the following corollary.

Corollary 9 If $f(z)$ in equation (1) be in the class $B_{\Sigma}(1)$, then

$$|a_3| \le 1.$$

Theorem 8 If $f \in B_{\Sigma}(\beta,\lambda)$, $0 \le \beta < 1$, $\lambda \ge 0$ and$\mu\epsilon\mathbb{R}$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{1-\beta}{5\lambda+1}|1-\mu| & \text{for } 3(3\lambda+1)|(1-\mu)| \ge 5\lambda+1, \\ \dfrac{1-\beta}{3(3\lambda+1)} & \text{for } 3(3\lambda+1)|(1-\mu)| \le 5\lambda+1. \end{cases}$$

(42)

Taking $\mu = 0$ and $\lambda = 0$ in Theorem 8, we obtain the following corollary.

Corollary 10 (Zaprawa, 2014) If $f(z)$ in equation (1) be in the class $B_{\Sigma}(\beta)$, $0 \le \beta < 1$, then

$$|a_3| \le 1-\beta.$$

IV. SUMMARY

In conclusion, coefficient estimates on $|a_2|$ and $|a_3|$ are obtained for some subclasses of bi-univalent functions. Furthermore, the upper bounds of the Fekete-Szegö functional are also obtained.

V. REFERENCES

Altinkaya, Ş. & Yalcin, S. (2017). The Feketa-Szegö problem for a general class of bi-univalent functions satisfying subordinate conditions. Sahand Communications in Mathematical Analysis, 5(1): 1-7.

Aouf, M. K., El-Ashwah, R. M. & Zayed, H. M. (2013). Fekete-Szegö inequalities for certain class of meromorphic functions. Journal of the Egyptian Mathematical Society, 21:197-200.

Duren, P. L. (1983). Univalent functions. Grundlehren der mathematicschen Wissenschanften, 259. New York: Springer-Verlag.

Janteng, A., Halim, S. A. & Darus, M. (2006). Fekete-Szegö problem for certain subclass of quasi-convex functions. Int. J. Contemp. Math. Sci., 1(1):45-51.

Murugusundaramoorthy, G. & Magesh, N. (2009). Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant. Bulletin of Mathematical Analysis and Applications, 1(3): 85-89.

Shanmugam, T. & Sivasubramanian, S. (2005). On the Fekete-Szegö problem for some subclasses of analytic functions. J. Inequal. Pure Appl. Math., 6(3)(2005): 1-15.

Zaprawa, P. (2014). Estimates of Initial Coefficients for Bi-Univalent Function. Abstract and Applied Analysis, 2014.