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A Homological Invariant of Certain Torsion Free Crystallographic Groups

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Several homological invariants namely the nonabelian tensor square, the exterior square and the Schur multiplier of groups have been of research interests by group theorists over the years. Besides, there are also some other homological invariants which can be deduced from these invariants, as example, the central subgroup of the nonabelian tensor square of a group G, known as $\nabla(G)$. The computations of the homological invariants of crystallographic groups strengthen the link between group theory with crystallography theory. In this paper, $\nabla(G)$ is determined for certain torsion free crystallographic groups focusing on those with cyclic point groups of order three and five.

Keywords: homological invariant, nonabelian tensor square, crystallographic

I. INTRODUCTION

A central subgroup of the nonabelian tensor square of a group G, namely $\nabla(G)$ is one of the important structures in the development of other homological invariants of that group including the exterior square and Schur multiplier. In some cases, the nonabelian tensor square itself can be determined by using $\nabla(G)$. Researches have been conducted on the computation of $\nabla(G)$ for various groups G including in (Abdul Ladi et al., 2017), (Mohd Idrus et al., 2015) and (Mohammad et al., 2016). A Bieberbach group is an extension of a free abelian group known as lattice subgroup with a finite point group.In this paper, $\nabla(G)$ is computed for certain torsion free crystallographic groups known as Bieberbach groups. Focus is given to the Bieberbach group with cyclic point groups of order three and five.

II. METHODS

Ellis and Leonard in (Ellis& Leonard, 1995) discovered the relation between the nonabelian tensor square of a group G with the commutator $[G,G^{\varphi}]$ as given in the following theorem.

Theorem 1.(Ellis & Leonard, 1995) *Let G be a group. The map $\sigma: G\otimes G \to [G,G^{\varphi}] \lhd \nu(G)$ defined by $\sigma(g\otimes h)=[g,h^{\varphi}]$ for all g,h in G is an isomorphism.*

By using the preceding theorem as a basis and the matrix presentation obtained from Crystallographic, Algorithms and Tables (CARAT) package (CARAT homepage), the polycyclic presentations of all Bieberbach groups with cyclic point group of order three, C_3, (up to dimension six) are constructed in (Mat Hassim, 2014).The polycyclic presentations of all non-isomorphic Bieberbach groups with point group C_3 are listed as follows:

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$$H_1(3) = \langle a, l_1, l_2, l_3 \mid a^3 = l_3, {}^a l_1 = l_1^{-1} l_2^{-1}, {}^a l_2 = l_1,$$
$${}^a l_3 = l_3, {}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_2} l_3 = l_3 \rangle,$$

$$H_2(4) = \langle a, l_1, l_2, l_3, l_4 \mid a^3 = l_3, {}^a l_1 = l_2^{-1},$$
$${}^a l_2 = l_1 l_2^{-1}, {}^a l_3 = l_3, {}^a l_4 = l_4,$$
$${}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_1} l_4 = l_4,$$
$${}^{l_2} l_3 = l_3, {}^{l_2} l_4 = l_4, {}^{l_3} l_4 = l_4 \rangle,$$

$$H_3(4) = \langle a, l_1, l_2, l_3, l_4 \mid a^3 = l_1, {}^a l_1 = l_1,$$
$${}^a l_2 = l_4^{-1}, {}^a l_3 = l_2, {}^a l_4 = l_3^{-1},$$
$${}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_1} l_4 = l_4,$$
$${}^{l_2} l_3 = l_3, {}^{l_2} l_4 = l_4, {}^{l_3} l_4 = l_4 \rangle,$$

$$H_4(5) = \langle a, l_1, l_2, l_3, l_4, l_5 \mid a^3 = l_5, {}^a l_1 = l_1^{-1} l_2^{-1},$$
$${}^a l_2 = l_1, {}^a l_3 = l_3^{-1} l_4^{-1}, {}^a l_4 = l_3, {}^a l_5 = l_5,$$
$${}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_1} l_4 = l_4, {}^{l_1} l_5 = l_5,$$
$${}^{l_2} l_3 = l_3, {}^{l_2} l_4 = l_4, {}^{l_2} l_5 = l_5, {}^{l_3} l_4 = l_4,$$
$${}^{l_3} l_5 = l_5, {}^{l_4} l_5 = l_5 \rangle,$$

$$H_5(5) = \langle a, l_1, l_2, l_3, l_4, l_5 \mid a^3 = l_3, {}^a l_1 = l_1^{-1} l_2^{-1},$$
$${}^a l_2 = l_1, {}^a l_3 = l_3, {}^a l_4 = l_4, {}^a l_5 = l_5,$$
$${}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_1} l_4 = l_4, {}^{l_1} l_5 = l_5,$$
$${}^{l_2} l_3 = l_3, {}^{l_2} l_4 = l_4, {}^{l_2} l_5 = l_5, {}^{l_3} l_4 = l_4,$$
$${}^{l_3} l_5 = l_5, {}^{l_4} l_5 = l_5 \rangle,$$

$$H_6(5) = \langle a, l_1, l_2, l_3, l_4, l_5 \mid a^3 = l_2, {}^a l_1 = l_1, {}^a l_2 = l_2,$$
$${}^a l_3 = l_5^{-1}, {}^a l_4 = l_3^{-1}, {}^a l_5 = l_4, {}^{l_1} l_2 = l_2,$$
$${}^{l_1} l_3 = l_3, {}^{l_1} l_4 = l_4, {}^{l_1} l_5 = l_5, {}^{l_2} l_3 = l_3,$$
$${}^{l_2} l_4 = l_4, {}^{l_2} l_5 = l_5, {}^{l_3} l_4 = l_4, {}^{l_3} l_5 = l_5,$$
$${}^{l_4} l_5 = l_5 \rangle,$$

$$H_7(6) = \langle a, l_1, l_2, l_3, l_4, l_5, l_6 \mid a^3 = l_5, {}^a l_1 = l_1^{-1} l_2^{-1},$$
$${}^a l_2 = l_1, {}^a l_3 = l_3^{-1} l_4, {}^a l_4 = l_3^{-1}, {}^a l_5 = l_5,$$
$${}^a l_6 = l_6, {}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_1} l_4 = l_4,$$
$${}^{l_1} l_5 = l_5, {}^{l_1} l_6 = l_6, {}^{l_2} l_3 = l_3, {}^{l_2} l_4 = l_4,$$
$${}^{l_2} l_5 = l_5, {}^{l_2} l_6 = l_6, {}^{l_3} l_4 = l_4, {}^{l_3} l_5 = l_5,$$
$${}^{l_3} l_6 = l_6, {}^{l_4} l_5 = l_5, {}^{l_4} l_6 = l_6, {}^{l_5} l_6 = l_6 \rangle,$$

$$H_8(6) = \langle a, l_1, l_2, l_3, l_4, l_5, l_6 \mid a^3 = l_1, {}^a l_1 = l_1,$$
$${}^a l_2 = l_2^{-1} l_3, {}^a l_3 = l_2^{-1}, {}^a l_4 = l_5^{-1},$$
$${}^a l_5 = l_6^{-1}, {}^a l_6 = l_4, {}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3,$$
$${}^{l_1} l_4 = l_4, {}^{l_1} l_5 = l_5, {}^{l_1} l_6 = l_6, {}^{l_2} l_3 = l_3,$$
$${}^{l_2} l_4 = l_4, {}^{l_2} l_5 = l_5, {}^{l_2} l_6 = l_6, {}^{l_3} l_4 = l_4,$$
$${}^{l_3} l_5 = l_5, {}^{l_3} l_6 = l_6, {}^{l_4} l_5 = l_5, {}^{l_4} l_6 = l_6,$$
$${}^{l_5} l_6 = l_6 \rangle,$$

$$H_9(6) = \langle a, l_1, l_2, l_3, l_4, l_5, l_6 \mid a^3 = l_3, {}^a l_1 = l_1^{-1} l_2^{-1},$$
$${}^a l_2 = l_1, {}^a l_3 = l_3, {}^a l_4 = l_4, {}^a l_5 = l_5,$$
$${}^a l_6 = l_6, {}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_1} l_4 = l_4,$$
$${}^{l_1} l_5 = l_5, {}^{l_1} l_6 = l_6, {}^{l_2} l_3 = l_3, {}^{l_2} l_4 = l_4,$$
$${}^{l_2} l_5 = l_5, {}^{l_2} l_6 = l_6, {}^{l_3} l_4 = l_4, {}^{l_3} l_5 = l_5,$$
$${}^{l_3} l_6 = l_6, {}^{l_4} l_5 = l_5, {}^{l_4} l_6 = l_6, {}^{l_5} l_6 = l_6 \rangle,$$

$$H_{10}(6) = \langle a, l_1, l_2, l_3, l_4, l_5, l_6 \mid a^3 = l_2, {}^a l_1 = l_1,$$
$${}^a l_2 = l_2, {}^a l_3 = l_3, {}^a l_4 = l_6^{-1}, {}^a l_5 = l_4^{-1},$$
$${}^a l_6 = l_5, {}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_1} l_4 = l_4,$$
$${}^{l_1} l_5 = l_5, {}^{l_1} l_6 = l_6, {}^{l_2} l_3 = l_3, {}^{l_2} l_4 = l_4,$$
$${}^{l_2} l_5 = l_5, {}^{l_2} l_6 = l_6, {}^{l_3} l_4 = l_4, {}^{l_3} l_5 = l_5,$$
$${}^{l_3} l_6 = l_6, {}^{l_4} l_5 = l_5, {}^{l_4} l_6 = l_6, {}^{l_5} l_6 = l_6 \rangle.$$

Based on (Mat Hassim, 2014), the Bieberbach groups $H_5(5)$ and $H_9(6)$ belong to the same family as $H_1(3)$. Hence, the properties of these two groups coincide with the properties of $H_1(3)$. Next, the derived subgroups and the abelianisations of the Bieberbach groups with point group C_3 are presented in the next two propositions.

(7)

Proposition 1. (Mat Hassim et al., 2014) *The derived subgroups of the Bieberbach groups with point group* C_3 *are given as follows :*

$$H_1(3)' = \langle l_1l_2^{-1}, l_1^{-2}l_2^{-1}\rangle \cong C_0^2,$$
$$H_2(4)' = \langle l_1^{-1}l_2^{-1}, l_1l_2^{-2}\rangle \cong C_0^2,$$
$$H_3(4)' = \langle l_2l_3^{-1}, l_2^{-1}l_4^{-1}\rangle \cong C_0^2,$$
$$H_4(5)' = \langle l_1l_2^{-1}, l_1^{-2}l_2^{-1}, l_3l_4^{-1}, l_3^{-2}l_4^{-1}\rangle \cong C_0^4,$$
$$H_6(5)' = \langle l_3^{-1}l_4^{-1}, l_4l_5^{-1}\rangle \cong C_0^2,$$
$$H_7(6)' = \langle l_1l_2^{-1}, l_1^{-2}l_2^{-1}, l_3^{-1}l_4^{-1}, l_3^{-2}l_4\rangle \cong C_0^4,$$
$$H_8(6)' = \langle l_2^{-1}l_3^{-1}, l_2^{-2}l_3, l_4^{-1}l_5^{-1}, l_4l_6^{-1}\rangle \cong C_0^4,$$
$$H_{10}(6)' = \langle l_4^{-1}l_5^{-1}, l_5l_6^{-1}\rangle \cong C_0^2.$$

Proposition 2. (Mat Hassim et al., 2014)*The abelianisation of the Bieberbach groups with point group C_3 are as in the following :*

$$H_1(3)^{ab} = \langle aH_1(3)', l_1H_1(3)'\rangle \cong C_0 \times C_3,$$
$$H_2(4)^{ab} = \langle aH_2(4)', l_1H_2(4)', l_4H_2(4)'\rangle \cong C_0 \times C_3 \times C_0,$$
$$H_3(4)^{ab} = \langle aH_3(4)', l_2H_3(4)'\rangle \cong C_0^2,$$
$$H_4(5)^{ab} = \langle aH_4(5)', l_1H_4(5)', l_3H_4(5)'\rangle \cong C_0 \times C_3 \times C_3,$$
$$H_6(5)^{ab} = \langle aH_6(5)', l_1H_6(5)', l_3H_6(5)'\rangle \cong C_0^3,$$
$$H_7(6)^{ab} = \langle aH_7(6)', l_1H_7(6)', l_3H_7(6)', l_6H_7(6)'\rangle \cong C_0 \times C_3 \times C_3 \times C_0,$$
$$H_8(6)^{ab} = \langle aH_8(6)', l_2H_8(6)', l_4H_8(6)'\rangle \cong C_0 \times C_3 \times C_0,$$
$$H_{10}(6)^{ab} = \langle aH_{10}(6)', l_1H_{10}(6)', l_3H_{10}(6)', l_4H_{10}(6)'\rangle \cong C_0^4.$$

In addition, the preliminary results from previous researches which are used in the computation of $\nabla(G)$ of the chosen Bieberbach groups are also included in this section.

Proposition 3. (Blyth et al., 2010) *Let G be a group such that G^{ab} is finitely generated. Assume that G^{ab} is the direct product of the cyclic groups $\langle x_iG'\rangle$, for $i = 1,\ldots,s$ and set $E(G)$ to be $\langle [x_i, x_j^{\varphi}] \mid i < j\rangle [G, G'^{\varphi}]$. Then the following hold:*

1. $\nabla(G)$ is generated by the elements of the set
$$\{[x_i, x_i^{\varphi}], [x_i, x_j^{\varphi}][x_j, x_i^{\varphi}] \mid 1 \le i < j \le s\};$$
2. $[G, G^{\varphi}] = \nabla(G)E(G)$.

Proposition 4. (Blyth et al., 2010; Blyth & Morse, 2009)*Let g and h be elements of G such that $[g,h] = 1$. Then in $\nu(G)$,*

1. $[g^n, h^{\varphi}] = [g, h^{\varphi}]^n = [g, (h^{\varphi})^n]$ for all integers n;
2. $[g^n, (h^m)^{\varphi}][h^m, (g^n)^{\varphi}] = ([g, h^{\varphi}][h, g^{\varphi}])^{nm}$;
3. $[g, h^{\varphi}]$ is in the center of $\nu(G)$

Proposition 5. (Masri, 2009) *Let G be a group and $a, b, c \in G$. Then, $[[a,b], c^{\varphi}] = [c, [a,b]^{\varphi}]^{-1}$ in $\nu(G)$.*

Proposition 6. (Masri, 2009) *Let G and H be groups and let $g \in G$. Suppose ϕ is a homomorphism from G onto H. If $\phi(g)$ has finite order then $|\phi(g)|$ divides $|g|$. Otherwise the order of $\phi(g)$ equals the order of g.*

Proposition 7. (Zomorodian, 2005) *Let A, B and C be abelian groups and C_0 is the infinite cyclic group. Consider the ordinary tensor product of two abelian groups. Then,*

1. $C_0 \otimes A \cong A$,
2. $C_0 \otimes C_0 \cong C_0$,
3. $C_n \otimes C_m \cong C_{gcd(n,m)}$, for $n, m \in \mathbb{Z}$, *and*
4. $A \otimes (B \times C) = (A \otimes B) \times (A \otimes C)$.

Theorem 2. (Brown et al., 1987) *Let G and H be groups such that there is an epimorphism $\epsilon : G \to H$. Then there exists an epimorphism*

$$\alpha : G \otimes G \to H \otimes H$$

defined by $\alpha(g \otimes h) = \epsilon(g) \otimes \epsilon(h)$.

III. RESULTS AND DISCUSSIONS

This section is divided into two parts; the first part is on the computation of $\nabla(G)$ of the Bieberbach groups with point group C_3, namely $H_i(j)$, while the second part consists the computation of $\nabla(G)$ of the Bieberbach groups with point group C_5, denoted as $N_i(j)$.

A. The Homological Invariant $\nabla(G)$ of the Bieberbach Groups with Point Group C_3

The computation of $\nabla(H_1(3))$ is presented in the following theorem.

Theorem 3. *Let $H_1(3)$ be a Bieberbach group with point group C_3 of dimension three. Then,*

$$\nabla(H_1(3)) = \langle a\otimes a, l_1\otimes l_1, (a\otimes l_1)(l_1\otimes a)\rangle \cong C_0\times C_3\times C_3.$$

Proof.

By Proposition 2, the abelianisation of $H_1(3)$ is generated by $aH_1(3)'$ and $l_1H_1(3)'$ of infinite order and order three, respectively. Then, based on Proposition 3 (i), $\nabla(H_1(3))$ is generated by $[a,a^{\varphi}]$, $[l_1,l_1^{\varphi}]$ and $[a,l_1^{\varphi}][l_1,a^{\varphi}]$. Since $\nabla(H_1(3))$ is a subgroup of $H_1(3)\otimes H_1(3)$ and the mapping $\sigma: H_1(3)\otimes H_1(3)\to[H_1(3),H_1(3)^{\varphi}]$ defined by $\sigma(g\otimes h)=[g,h^{\varphi}]$ for all $g,h\in H_1(3)$ is anisomorphism, thus

$$\nabla(H_1(3)) = \langle a\otimes a, l_1\otimes l_1, (a\otimes l_1)(l_1\otimes a)\rangle.$$

Next, the order of each of the generators of $\nabla(H_1(3))$ is computed. By Proposition 1, l_1^3 is in $H_1(3)'$. Hence,by the identities of commutator, $[l_1,l_1^{\varphi}]^9=[l_1^3,(l_1^3)^{\varphi}]=1$. Therefore, $[l_1,l_1^{\varphi}]$ has order dividing 9. Suppose the order of $[l_1,l_1^{\varphi}]$ is 9. Then, there is no positive integer r smaller than 9, such that $[l_1,l_1^{\varphi}]^r=1$. However, since $l_1^3\in H_1(3)'$, then by Proposition 5, $[l_1,(l_1^3)^{\varphi}]=[l_1^3,l_1^{\varphi}]^{-1}$. This implies that $[l_1,(l_1^3)^{\varphi}][l_1^3,l_1^{\varphi}]=1$. By Proposition 4(i), $[l_1,l_1^{\varphi}]^3[l_1,l_1^{\varphi}]^3=1$. Hence, it is found that $[l_1,l_1^{\varphi}]^6=1$, a contradiction. Thus, the order of $[l_1,l_1^{\varphi}]$ cannot be 9. Since the greatest common divisor of 6 and 9 is 3, then, the order of $[l_1,l_1^{\varphi}]$ is 3.

Next, the order of $[a,l_1^{\varphi}][l_1,a^{\varphi}]$ is also 3 based on the commutator identities and Proposition 5;

$$\begin{aligned}([a,l_1^{\varphi}][l_1,a^{\varphi}])^3 &= [a,(l_1^3)^{\varphi}][l_1^3,a^{\varphi}]\\ &= [a,(l_1^3)^{\varphi}][a,(l_1^3)^{\varphi}]^{-1}\\ &= 1.\end{aligned}$$

The abelianisation of $H_1(3)$ is denoted by $H_1(3)^{ab}$ with natural homomorphism $\epsilon: H_1(3)\to H_1(3)^{ab}$. Since $H_1(3)^{ab}$ is finitely generated, then its nonabelian tensor square is simply the ordinary tensor product of $H_1(3)^{ab}\cong C_0\times C_3$. By Proposition 7,

$$\begin{aligned}H_1(3)^{ab}\otimes H_1(3)^{ab} &\cong (C_0\times C_3)\otimes(C_0\times C_3)\\ &\cong C_0\times C_3\times C_3\times C_3.\end{aligned}$$

Then, Proposition 2 provides that $H_1(3)^{ab}$ is generated by $\epsilon(a)$ and $\epsilon(l_1)$ of infinite order and order three, respectively. Again, by Proposition 7, $\langle\epsilon(a)\otimes\epsilon(a)\rangle\cong C_0$, $\langle\epsilon(a)\otimes\epsilon(l_1)\rangle\cong C_3$, $\langle\epsilon(l_1)\otimes\epsilon(a)\rangle\cong C_3$, and $\langle\epsilon(l_1)\otimes\epsilon(l_1)\rangle\cong C_3$. By Theorem 2, there is a natural epimorphism

$$\alpha: H_1(3)\otimes H_1(3)\to H_1(3)^{ab}\otimes H_1(3)^{ab}.$$

Therefore, the image $\alpha(a\otimes a)=\epsilon(a)\otimes\epsilon(a)$ has infinite order. Thus by Proposition 6, $a\otimes a$ has also infinite order. Finally, the desired result is obtained, which is

$$\nabla(H_1(3)) = \langle a\otimes a, l_1\otimes l_1, (a\otimes l_1)(l_1\otimes a)\rangle \cong C_0\times C_3\times C_3. \quad \square$$

Using similar method as above, $\nabla(H_i(j))$ can be computed for other Bieberbach groups with point group C_3. The results are summarized in the following table:

Table 1: The homological functor $\nabla(H_i(j))$

$H_i(j)$	$\nabla(H_i(j))$	$\cong$
$H_1(3)$	$\langle a\otimes a, l_1\otimes l_1, (a\otimes l_1)(l_1\otimes a)\rangle$	$C_0\times C_3 \times C_3$
$H_2(4)$	$\langle a\otimes a, l_1\otimes l_1, l_4\otimes l_4, (a\otimes l_1)(l_1\otimes a), (a\otimes l_4)(l_4\otimes a), (l_1\otimes l_4)(l_4\otimes l_1)\rangle$	$C_0^3\times C_3^3$
$H_3(4)$	$\langle a\otimes a, l_2\otimes l_2, (a\otimes l_2)(l_2\otimes a)\rangle$	C_0^3
$H_4(5)$	$\langle a\otimes a, l_1\otimes l_1, l_3\otimes l_3, (a\otimes l_1)(l_1\otimes a), (a\otimes l_3)(l_3\otimes a), (l_1\otimes l_3)(l_3\otimes l_1)\rangle$	$C_0\times C_3^5$
$H_6(5)$	$\langle a\otimes a, l_1\otimes l_1, l_3\otimes l_3, (a\otimes l_1)(l_1\otimes a), (a\otimes l_3)(l_3\otimes a), (l_1\otimes l_3)(l_3\otimes l_1)\rangle$	C_0^6
$H_7(6)$	$\langle a\otimes a, l_1\otimes l_1, l_3\otimes l_3, l_6\otimes l_6, (a\otimes l_1)(l_1\otimes a), (a\otimes l_3)(l_3\otimes a), (a\otimes l_6)(l_6\otimes a), (l_1\otimes l_3)(l_3\otimes l_1), (l_1\otimes l_6)(l_6\otimes l_1), (l_3\otimes l_6)(l_6\otimes l_3)\rangle$	$C_0^3\times C_3^7$
$H_8(6)$	$\langle a\otimes a, l_2\otimes l_2, l_4\otimes l_4, (a\otimes l_2)(l_2\otimes a), (a\otimes l_4)(l_4\otimes a), (l_2\otimes l_4)(l_4\otimes l_2)\rangle$	$C_0^3\times C_3^3$
$H_{10}(6)$	$\langle a\otimes a, l_1\otimes l_1, l_3\otimes l_3, l_4\otimes l_4, (a\otimes l_1)(l_1\otimes a), (a\otimes l_3)(l_3\otimes a), (a\otimes l_4)(l_4\otimes a), (l_1\otimes l_3)(l_3\otimes l_1), (l_1\otimes l_4)(l_4\otimes l_1), (l_3\otimes l_4)(l_4\otimes l_3)\rangle$	C_0^{10}

B. The Homological Invariant $\nabla(G)$ of the Bieberbach Groups with Point Group C_5

The polycyclic presentations of all non-isomorphic Bieberbach groups with cyclic point group of order five, up to dimension six are given in this subsection. The CARAT package shows that there are three Bieberbach groups with cyclic point group of order 5, C_5, up to dimension six. Using the definition of the polycyclic presentation, the constructed polycyclic presentations of these Bieberbach groups are given as follows:

$$\begin{aligned} N_1(5) = \langle a, l_1, l_2, l_3, l_4, l_5 \mid\ & a^5 = l_5, {}^a l_1 = l_4^{-1}, \\ & {}^a l_2 = l_2^{-1} l_3^{-1}, {}^a l_3 = l_1, {}^a l_4 = l_1 l_2^{-1}, \\ & {}^a l_5 = l_5, {}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_1} l_4 = l_4, \\ & {}^{l_1} l_5 = l_5, {}^{l_2} l_3 = l_3, {}^{l_2} l_4 = l_4, {}^{l_2} l_5 = l_5, \\ & {}^{l_3} l_4 = l_4, {}^{l_3} l_5 = l_5, {}^{l_4} l_5 = l_5 \rangle, \end{aligned}$$

$$\begin{aligned} N_2(6) = \langle a, l_1, l_2, l_3, l_4, l_5, l_6 \mid\ & a^5 = l_5, {}^a l_1 = l_4, \\ & {}^a l_2 = l_1 l_2^{-1} l_3^{-1}, {}^a l_3 = l_1^{-1} l_2, {}^a l_4 = l_2^{-1}, \\ & {}^a l_5 = l_5, {}^a l_6 = l_6, {}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, \\ & {}^{l_1} l_4 = l_4, {}^{l_1} l_5 = l_5, {}^{l_1} l_6 = l_6, {}^{l_2} l_3 = l_3, \\ & {}^{l_2} l_4 = l_4, {}^{l_2} l_5 = l_5, {}^{l_2} l_6 = l_6, {}^{l_3} l_4 = l_4, \\ & {}^{l_3} l_5 = l_5, {}^{l_3} l_6 = l_6, {}^{l_4} l_5 = l_5, {}^{l_4} l_6 = l_6, \\ & {}^{l_5} l_6 = l_6 \rangle, \end{aligned}$$

$$\begin{aligned} N_3(6) = \langle a, l_1, l_2, l_3, l_4, l_5, l_6 \mid\ & a^5 = l_1, {}^a l_1 = l_1, \\ & {}^a l_2 = l_4, {}^a l_3 = l_5^{-1}, {}^a l_4 = l_6, {}^a l_5 = l_2^{-1}, \\ & {}^a l_6 = l_3, {}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_1} l_4 = l_4, \\ & {}^{l_1} l_5 = l_5, {}^{l_1} l_6 = l_6, {}^{l_2} l_3 = l_3, {}^{l_2} l_4 = l_4, \\ & {}^{l_2} l_5 = l_5, {}^{l_2} l_6 = l_6, {}^{l_3} l_4 = l_4, {}^{l_3} l_5 = l_5, \\ & {}^{l_3} l_6 = l_6, {}^{l_4} l_5 = l_5, {}^{l_4} l_6 = l_6, {}^{l_5} l_6 = l_6 \rangle. \end{aligned}$$

Next, the derived subgroups and abelianisation of the Bieberbach groups with point group C_5 are presented in the following two lemmas.

Lemma 1. *The derived subgroups of the Bieberbach groups with point group C_5 are given as :*

$$N_1(5)' = \langle l_1^{-1} l_4^{-1}, l_2^{-2} l_3^{-1}, l_1 l_3^{-1}, l_1 l_2^{-1} l_4^{-1} \rangle \cong {C_0}^4,$$
$$N_2(6)' = \langle l_1^{-1} l_4, l_1 l_2^{-2} l_3^{-1}, l_1^{-1} l_2 l_3^{-1}, l_2^{-1} l_4^{-1} \rangle \cong C_0^4,$$
$$N_3(6)' = \langle l_2^{-1} l_4, l_3^{-1} l_5^{-1}, l_4^{-1} l_6, l_3 l_6^{-1} \rangle \cong C_0^4.$$

Proof.

The Bieberbach group $N_1(5)$ is generated by elements a, l_1, l_2, l_3, l_4 and l_5 where $[a,l_1] = l_1^{-1}l_4^{-1}$, $[a,l_2] = l_2^{-2}l_3^{-1}$, $[a,l_3] = l_1l_3^{-1}$, $[a,l_4] = l_1l_2^{-1}l_4^{-1}$, $[a,l_5] = 1$ and $[l_i,l_j] = 1$ for all $1 \le i < j \le 5$. Then,

$$N_1(5)' = \langle l_1^{-1}l_4^{-1}, l_2^{-2}l_3^{-1}, l_1l_3^{-1}, l_1l_2^{-1}l_4^{-1} \rangle$$

and is isomorphic to C_0^4 since $N_1(5)$ is torsionfree. Using similar method, the derived subgroups of $N_2(6)$ and $N_3(6)$ are found to be

$$N_2(6)' = \langle l_1^{-1}l_4, l_1l_2^{-2}l_3^{-1}, l_1^{-1}l_2l_3^{-1}, l_2^{-1}l_4^{-1} \rangle \cong C_0^4,$$
$$N_3(6)' = \langle l_2^{-1}l_4, l_3^{-1}l_5^{-1}, l_4^{-1}l_6, l_3l_6^{-1} \rangle \cong C_0^4. \quad \square$$

Lemma 2. *The abelianisation of the Bieberbach groups with point group C_5 are :*

$$N_1(5)^{ab} = \langle aN_1(5)', l_1N_1(5)' \rangle \cong C_0 \times C_5,$$
$$N_2(6)^{ab} = \langle aN_2(6)', l_1N_2(6)', l_6N_2(6)' \rangle \cong C_0 \times C_5 \times C_0,$$
$$N_3(6)^{ab} = \langle aN_3(6)', l_2N_3(6)' \rangle \cong C_0^2.$$

Proof.

The abelianisation $N_1(5)^{ab}$ is defined as $N_1(5)^{ab} = N_1(5)/N_1(5)'$. Hence, it is generated by $aN_1(5)'$, $l_1N_1(5)'$, $l_2N_1(5)'$, $l_3N_1(5)'$, $l_4N_1(5)'$ and $l_5N_1(5)'$. However, the generators are not independent. By the relations of $N_1(5)$, $a^5 = l_5$. Hence $aN_1(5)' \cap l_5N_1(5)'$ is not trivial, and $aN_1(5)' = l_5N_1(5)'$. Next, $l_1N_1(5)' = l_3N_1(5)'$ and $l_4N_1(5)' = (l_1N_1(5)')^{-1}$ since $l_1l_3^{-1}, l_1^{-1}l_4^{-1} \in N_1(5)'$. Besides, $l_1(l_2l_4)^{-1}$ is also in $N_1(5)'$. Thus, $l_1N_1(5)' = (l_2l_4)N_1(5)'$. By properties of a factor group $l_1N_1(5)' = l_2N_1(5)'l_4N_1(5)'$. Then, since $l_4N_1(5)' = (l_1N_1(5)')^{-1}$, it is shown that $l_2N_1(5)' = (l_1N_1(5)')^2$. Therefore, the independent generators of $N_1(5)^{ab}$ are $aN_1(5)'$ and $l_1N_1(5)'$.

By the relations of $N_1(5)$, for any integer r, a^r is not in $N_1(5)'$. Since $N_1(5)'$ is generated by elements of infinite order, then the order of $aN_1(5)'$ is infinite. Meanwhile, it can be shown that $l_1^5 = (l_1l_3^{-1})(l_1^{-1}l_4^{-1})^{-2}(l_1l_2^{-1}l_4^{-1})^2(l_2^{-2}l_3^{-1})^{-1}$ is in $N_1(5)'$ since $l_1l_3^{-1}, l_1^{-1}l_4^{-1}, l_1l_2^{-1}l_4^{-1}, l_2^{-2}l_3^{-1} \in N_1(5)'$.

This implies that $l_1N_1(5)'$ has order 5. Therefore, $N_1(5)^{ab} = \langle aN_1(5)', l_1N_1(5)' \rangle \cong C_0 \times C_5$. Using similar method, the abelianisation of $N_2(6)$ and $N_3(6)$ are found to be

$$N_2(6)^{ab} = \langle aN_2(6)', l_1N_2(6)', l_6N_2(6)' \rangle \cong C_0 \times C_5 \times C_0,$$
$$N_3(6)^{ab} = \langle aN_3(6)', l_2N_3(6)' \rangle \cong C_0^2. \quad \square$$

Theorem 4. *Let $N_1(5)$ be a Bieberbach group with point group C_5 of dimension five. Then,* $\nabla(N_1(5)) = \langle a \otimes a, l_1 \otimes l_1, (a \otimes l_1)(l_1 \otimes a) \rangle \cong C_0 \times C_5^2$.

Proof.

Based on Theorem 1, Proposition 3(i) and Lemma 2, $\nabla(N_1(5))$ is generated by $a \otimes a$, $l_1 \otimes l_1$ and $(a \otimes l_1)(l_1 \otimes a)$. The abelianisation of $N_1(5)$ is denoted by $N_1(5)^{ab}$ with natural homomorphism $\epsilon : N_1(5) \to N_1(5)^{ab}$. Since $N_1(5)^{ab} \cong C_0 \times C_5$, then by Proposition 7,

$$\begin{aligned} N_1(5)^{ab} \otimes N_1(5)^{ab} &\cong (C_0 \times C_5) \otimes (C_0 \times C_5) \\ &\cong (C_0 \otimes (C_0 \times C_5)) \times (C_5 \otimes (C_0 \times C_5)) \\ &\cong C_0 \times C_5 \times C_5 \times C_5. \end{aligned}$$

Based on Lemma 2, $N_1(5)^{ab}$ is generated by $\epsilon(a)$ and $\epsilon(l_1)$ of order infinity and 5, respectively. Again, by Proposition 7, $\langle \epsilon(a) \otimes \epsilon(a) \rangle \cong C_0$, $\langle \epsilon(a) \otimes \epsilon(l_1) \rangle \cong C_5$, $\langle \epsilon(l_1) \otimes \epsilon(a) \rangle \cong C_5$, $\langle \epsilon(l_1) \otimes \epsilon(l_1) \rangle \cong C_5$. By Theorem 2, there is a natural epimorphism

$$\alpha : N_1(5) \otimes N_1(5) \to N_1(5)^{ab} \otimes N_1(5)^{ab}.$$

Therefore, the image $\alpha(a \otimes a) = \epsilon(a) \otimes \epsilon(a)$ has infinite order. Thus by Proposition 6, $a \otimes a$ has infinite order.

Since l_1^5 is in $N_1(5)'$, then $[l_1^5,(l_1^5)^{\varphi}]=1$. Thus, by applying Proposition 4(i), $[l_1,l_1^{\varphi}]^{25}$ is trivial,

which implies that $[l_1,l_1^{\varphi}]$ has order dividing 25. Suppose the order of $[l_1,l_1^{\varphi}]$ is 25, that is the smallest integer m such that $[l_1,l_1^{\varphi}]^m=1$ is 25. This is a contradiction since by Propositions 4(i) and 5, $[l_1,l_1^{\varphi}]^{10}=1$ due to the fact that $[l_1^5,l_1^{\varphi}]=[l_1,(l_1^5)^{\varphi}]^{-1}$. Since the greatest common divisor of 25 and 10 is 5, then the order of $[l_1,l_1^{\varphi}]$ is 5. By Theorem 1, $l_1\otimes l_1$ has order 5.

Next, the order of $(a\otimes l_1)(l_1\otimes a)$ can be shown to be 5. By the identities of comutator, $([a,l_1^{\varphi}][l_1,a^{\varphi}])^5=[a,(l_1^5)^{\varphi}][l_1^5,a^{\varphi}]$. Again, since $l_1^5\in N_1(5)'$, then $[l_1^5,a^{\varphi}]=[a,(l_1^5)^{\varphi}]^{-1}$. Hence, $([a,l_1^{\varphi}][l_1,a^{\varphi}])^5=1$. Then, by Theorem 1, the order of $(a\otimes l_1)(l_1\otimes a)$ is 5. Therefore,

$$\begin{aligned}\nabla(N_1(5)) &= \langle a\otimes a, l_1\otimes l_1, (a\otimes l_1)(l_1\otimes a)\rangle\\ &\cong C_0\times C_5^2. \ \square\end{aligned}$$

Similar method is applied to the other Bieberbach groups with point group C_5 and the results for all $\nabla(N_i(j))$ are given in the following table.

Table 2: The homological functor $\nabla(N_i(j))$

$N_i(j)$	$\nabla(N_i(j))$	$\cong$
$N_1(5)$	$\langle a\otimes a, l_1\otimes l_1,$ $(a\otimes l_1)(l_1\otimes a)\rangle$	$C_0\times C_3$
$N_2(6)$	$\langle a\otimes a, l_1\otimes l_1, l_6\otimes l_6,$ $(a\otimes l_1)(l_1\otimes a),$ $(a\otimes l_6)(l_6\otimes a),$ $(l_1\otimes l_6)(l_6\otimes l_1)\rangle$	$C_0^3\times C_5^3$
$N_3(6)$	$\langle a\otimes a, l_2\otimes l_2,$ $(a\otimes l_2)(l_2\otimes a)\rangle$	C_0^3

IV. SUMMARY

In this paper, the computation of homological invariant $\nabla(G)$ for all non-isomorphic Bieberbach groups with cyclic point groups of order three and five, up to dimension six, are presented. The results are summarized in Table 1 and Table 2. Since $\nabla(G)$ is a central subgroup of the nonabelian tensor square, then all the results are abelian and isomorphic to product of several copies of cyclic groups.

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