ASM Sc. J., 13, 2020
https://doi.org/10.32802/asmscj.2020.sm26(5.6)

Bounds of the Coefficient Estimates for a Subclass of Bi-Univalent Functions

Nurdiana Nurali[1,*], Aini Janteng[2] and Rashidah Omar[3]

[1]*Faculty of Science and Natural Resources, Universiti Malaysia Sabah, 88400 Kota Kinabalu, Sabah, Malaysia*

[2]*Faculty of Science and Natural Resources, Universiti Malaysia Sabah, 88400 Kota Kinabalu, Sabah, Malaysia*

[3]*Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA Cawangan Sabah, 88997 Kota Kinabalu, Sabah, Malaysia*

In this paper, subclass of the class ς of bi-univalent function is introduced using subordination. For functions in the class ς, theestimates on the Taylor-Maclaurin coefficients and upper bound of the Fekete-Szegö functional are obtained.

Keywords: analytic, bi-univalent, subordination, Fekete-Szegö

I. INTRODUCTION

Let A denote the class of functions in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and normalized by $f(0) = 0$ and $f'(0) = 1$in the open unit disk $\Delta := \{z: z \in C \text{ and } |z| < 1\}$. Let$S$denote the subclass of A. Functions in $S$are univalent.

Let S_S^* be the subclass belonging to S in the form (1) satisfying

$$Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0 \tag{2}$$

for $z \in \Delta$. Sakaguchi (1959) introduced functions of starlike with respect to symmetric points as stated in (2). Das & Singh (1977) then introduced another class C_S namely, convex functions with respect to symmetric points and satisfying the condition

$$Re\left(\frac{(zf'(z))'}{(f(z) - f(-z))'}\right) > 0 \tag{3}$$

for $z \in \Delta$.

If $f \in S$, then there exists an inverse function, f^{-1} which is also univalent in Δ.A function $f \in$ A is called bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Thus, ς denote the class of bi-univalent functions defined in Δ. All bi-univalent functions have an inverse function with the Taylor series in the form

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - \cdots \tag{4}$$

Many earlier researches published papers concerning bi-univalent functions. These researchers focused on problems connected with coefficients. This research was initiated by Lewin in 1967. Brannan & Taha (1986) instigated certain subclasses of bi-univalent functions including of starlike, strongly starlike and convex functions which have the similarities with subclasses of univalent functions.

A function f is said to be subordinate to function g if both functions are analytic in Δ, and if there is exists function w, defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. This subordination can be expressed in the form of $f(z) \prec g(z)$.

*Corresponding author's e-mail: nurdiana4492@yahoo.com

Ma & Minda (1994) combined diverse subclasses of convex and starlike functions. They cogitate an analytic function Φ with positive real part in Δ, where $\Phi(0) = 1$, $\Phi'(0) > 0$ that maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. This series of expansion for function Φ can be demonstrated in the form of

$$\Phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, (B_1 > 0) \quad (5)$$

The classes of Ma-Mindastarlike and Ma-Minda convex functions consist of functions $f \in A$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \Phi(z)$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \Phi(z)$$

respectively. Both f and f^{-1} must be Ma-Minda starlike or convex, autonomously in order for function f to be bi-starlike or bi-convex of Ma-Minda type. $ST_{\varsigma}(\Phi)$ denote the class for bi-starlike of Ma-Minda type and$CV_{\varsigma}(\Phi)$denote the class of bi-convex of Ma-Minda type.

In modern works, several authors such as Lashin (2016), Crisan (2013) and Ali et al.(2012) introduced new subclasses of class $\varsigma$and obtained the estimates of the initial coefficients of the Taylor-Maclaurin for functions belonging to these classes.

In this paper, approximations of the initial coefficients for bi-univalent of Ma-Mindastarlike and convex functions were studied. Further, the upper bound of the Fekete-Szegö functional also studied.

II. METHODS

Definition 1. A function $f(z) \in \varsigma$ given by (1) is said to be in a class $\mathcal{K}_{\varsigma}(\Phi, \alpha)$ with $0 < \alpha \leq 1$ if the following conditions are satisfied:

$$\frac{2\left(zf'(z) + \alpha z^2 f''(z)\right)}{(1-\alpha)(f(z) - f(-z)) + \alpha z(f(z) - f(-z))'} \prec \Phi(z) \quad (6)$$

and

$$\frac{2\left(wg'(w) + \alpha w^2 g''(w)\right)}{(1-\alpha)(g(w) - g(-w)) + \alpha w(g(w) - g(-w))'} \prec \Phi(w) \quad (7)$$

and the function $g = f^{-1}(w)$ is given by (4).

We first declare the well-known lemmas that will be used to acquire the upper bounds of the initial coefficients and the Fekete-Szegö functional for function $f \in \mathcal{K}_{\varsigma}(\Phi, \alpha)$.

Lemma 2.(Duren, 1983)*If* $p \in P$ *then* $|p_k| \leq 2$ *for each* k, *where* P *is the family of all functions* p *analytic in* Δ, $Re(p(z)) > 0$

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

for $z \in \Delta$.

Lemma 3. (Zaprawa, 2014) *Let* $k, l \in \mathbb{R}$ *and* $z_1, z_2 \in \mathbb{C}$. *If* $|z_1| < R$ *and* $|z_2| < R$ *then,*

$$|(k+l)z_1 + (k-l)z_2| \leq \begin{cases} 2|k|R & for\ |k| \geq |l| \\ 2|l|R & for\ |k| \leq |l| \end{cases} \quad (8)$$

III. RESULTS

For functions $\mathcal{K}_{\varsigma}(\Phi, \alpha)$, the following coefficient estimates are obtained as given by Theorem 4.

Theorem 4. *Let* f *given by (1) be in the class* $\mathcal{K}_{\varsigma}(\Phi, \alpha)$ *where* $0 < \alpha \leq 1$. *Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2|B_1^2(1+2\alpha) + 2(1+\alpha)^2(B_1 - B_2)|}} \quad (9)$$

and

$$|a_3| \leq \frac{1}{2}B_1\left(\frac{1}{(1+2\alpha)} + \frac{1}{2(1+\alpha)^2}B_1\right) \quad (10)$$

Proof. For $f \in \mathcal{K}_{\varsigma}(\Phi, \alpha)$, $g = f^{-1}$, there exist analytic functions $u, v: D \to D$ with $u(0) = v(0) = 0$, satisfying

$$\frac{2\left(zf'(z) + \alpha z^2 f''(z)\right)}{(1-\alpha)(f(z) - f(-z)) + \alpha z(f(z) - f(-z))'} = \Phi[u(z)] \quad (11)$$

and

$$\frac{2\left(wg'(w) + \alpha w^2 g''(w)\right)}{(1-\alpha)(g(w) - g(-w)) + \alpha w(g(w) - g(-w))'} = \Phi[v(w)] \quad (12)$$

The functions b and c are defined as:

$$b(z) := \frac{1+u(z)}{1-u(z)} = 1 + b_1 z + b_2 z^2 + \cdots \tag{13}$$

and

$$c(z) := \frac{1+v(z)}{1-v(z)} = 1 + c_1 z + c_2 z^2 + \cdots \tag{14}$$

or it is equivalent to

$$u(z) = \frac{1}{2}\left[b_1 z + \left(b_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right] \tag{15}$$

and

$$v(z) = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right] \tag{16}$$

Functions b and c are analytic in Δ with $b(0) = 1 = c(0)$. Since $b, c\colon \Delta \to \Delta$, the functions b and c have positive real parts in Δ and $|b_i| \le 2$ and $|c_i| \le 2$ for $i = 1,2$.

From (5) and (11)-(14), we obtain

$$\frac{2\left(zf'(z) + \alpha z^2 f''(z)\right)}{(1-\alpha)(f(z) - f(-z)) + \alpha z(f(z) - f(-z))'}$$
$$= 1 + \frac{1}{2}B_1 b_1 z + \left[\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2\right] z^2 + \cdots \tag{17}$$

and

$$\frac{2\left(wg'(w) + \alpha w^2 g''(w)\right)}{(1-\alpha)\big(g(w) - g(-w)\big) + \alpha w\big(g(w) - g(-w)\big)'}$$
$$= 1 + \frac{1}{2}B_1 c_1 w + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right] w^2 + \cdots \tag{18}$$

(17) and (18) will gives us

$$2(1+\alpha)a_2 = \frac{1}{2}B_1 b_1 \tag{19}$$

$$2(1+2\alpha)a_3 = \frac{1}{2}B_1 b_2 - \frac{1}{4}B_1 b_1^2 + \frac{1}{4}B_2 b_1^2 \tag{20}$$

$$-2(1+\alpha)a_2 = \frac{1}{2}B_1 c_1 \tag{21}$$

and

$$2(1+2\alpha)(2a_2^2 - a_3) = \frac{1}{2}B_1 c_2 - \frac{1}{4}B_1 c_1^2 + \frac{1}{4}B_2 c_1^2 \tag{22}$$

From (19) and (21), we get

$$b_1 = -c_1 \tag{23}$$

and

$$8(1+\alpha)^2 a_2^2 = \frac{1}{4}B_1^2(b_1^2 + c_1^2) \tag{24}$$

By considering (20), (22) and (24), we obtain

$$a_2^2 = \frac{B_1^3(b_2 + c_2)}{8B_1^2(1+2\alpha) + 16(1+\alpha)^2(B_1 - B_2)} \tag{25}$$

By applying triangle inequality and Lemma 2 for the coefficients b_2 and c_2 into equation (25), we finally get:

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{2|B_1^2(1+2\alpha) + 2(1+\alpha)^2(B_1 - B_2)|}}$$

This gives the bound on $|a_2|$ as stated in (9).

Next, we apply (20) and (22) in order to find the bound on $|a_3|$, we will get

$$a_3 = a_2^2 + \frac{B_1(b_2 - c_2)}{8(1+2\alpha)} \tag{26}$$

By replacing a_2^2 from (25) into (26), we obtain

$$a_3 = \frac{B_1^2(b_1^2 + c_1^2)}{32(1+\alpha)^2} + \frac{B_1(b_2 - c_2)}{8(1+2\alpha)} \tag{27}$$

Lastly, by applying triangle inequality and Lemma 3 for the coefficients b_1, b_2, c_1 and c_2 in(27), we obtain:

$$|a_3| \le \frac{1}{2}B_1\left(\frac{1}{(1+2\alpha)} + \frac{1}{2(1+\alpha)^2}B_1\right)$$

Theorem 4 is completely proven.

The nexttheorem concerning the Fekete-Szegö inequality for $\mathcal{K}_\varsigma(\Phi, \alpha)$ will be formulated as shown in Theorem 5.

Theorem 5. *Let f given by (1) be in the class $\mathcal{K}_\varsigma(\Phi, \alpha)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{B_1}{2(1+2\alpha)} \\ for\ \ |1-\mu| \le \mathcal{T}(\alpha) \\ \dfrac{B_1^3|1-\mu|}{|2B_1^2(1+2\alpha) + 4(1+\alpha)^2(B_1 - B_2)|} \\ for\ \ |1-\mu| \ge \mathcal{T}(\alpha) \end{cases} \tag{28}$$

where

$$\mathcal{T}(\alpha) = \frac{1}{(1+2\alpha)} \cdot \left|(1+2\alpha) + 2(1+\alpha)^2\left(\frac{B_1 - B_2}{B_1^2}\right)\right|$$

Proof. Let f given by (1) be in the class $\mathcal{K}_{\varsigma}(\Phi,\alpha)$ and $\mu \in \mathbb{R}$.

By using the definition of Fekete-Szego, $a_3-\mu a_2^2$ and applying equation (26), we obtain

$$a_3-\mu a_2^2=\left[a_2^2+\frac{B_1(b_2-c_2)}{8(1+2\alpha)}\right]-\mu a_2^2 \qquad (29)$$

It follows that,

$$a_3-\mu a_2^2=B_1\left(\left[h(\mu)+\frac{1}{8(1+2\alpha)}\right]b_2 + \left[h(\mu)-\frac{1}{8(1+2\alpha)}\right]c_2\right) \qquad (30)$$

where

$$h(\mu)=\frac{B_1^2(1-\mu)}{8B_1^2(1+2\alpha)+16(1+\alpha)^2(B_1-B_2)}$$

By applying Lemma 2 and Lemma 3 into equation (30), we can conclude that

$$|a_3-\mu a_2^2| \le B_1\begin{cases} 2|h(\mu)|(2) & for \quad |h(\mu)|\ge \dfrac{1}{8(1+2\alpha)} \\ 2\left|\dfrac{1}{8(1+2\alpha)}\right|(2) & for \quad |h(\mu)|\le \dfrac{1}{8(1+2\alpha)} \end{cases}$$

$$=\begin{cases} 4B_1|h(\mu)| & for \quad |h(\mu)|\ge \dfrac{1}{8(1+2\alpha)} \\ \dfrac{B_1}{2(1+2\alpha)} & for \quad |h(\mu)|\le \dfrac{1}{8(1+2\alpha)} \end{cases}$$

Finally, Theorem 5 is proven.

Taking $\mu=1$ in Theorem 5, we obtain the following corollary.

Corollary 6. If $f(z)$ is given by equation (1) be in the class $\mathcal{K}_{\varsigma}(\Phi,\alpha)$, then

$$|a_3-a_2^2|\le \frac{B_1}{2(1+2\alpha)}$$

IV. SUMMARY

Predominantly, the subclass of bi-univalent functions, $\mathcal{K}_{\varsigma}(\Phi,\alpha)$, is proposed using subordination. Then, the upper bound for the coefficients estimates and the Fekete-Szegö functional are obtained for function in the class $\mathcal{K}_{\varsigma}(\Phi,\alpha)$.

V. REFERENCES

Ali, R.M., Lee, S.K., Ravichandran, V. & Supramaniam, S. (2012). Coefficient estimates for bi-univalent Ma-Mindastarlike and convex functions. Applied Mathematics Letters, 25, 344-351.

Brannan, D.A. & Clunie, J.G. (1980). Aspects of Contemporary Complex Analysis. Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham, July 1-20, 1979, Academy Press, London, New York.

Brannan, D.A., Clunie, J.& Kirwan, W.E. (1970). Coefficient estimates for a class of starlike functions. Canadian Mathematical Society, 22, 476-485.

Brannan, D.A.& Taha, T.S. (1987). On some classes of bi-univalent functions. Studia Universitatis Babes-Bolyai Mathematica, 31 (2), 70-77.

Bulut, S. (2013). Coefficient estimates for a class of analytic and bi-univalent functions. Novi Sad J. Math., 43 (2), 59-65.

Crisan, O. (2013). Coefficient Estimates for Certain Subclasses of Bi-Univalent Functions. Gen. Math. Notes, 16, 93-102.

Das, R.N.& Singh, P. (1977). On Subclass of Schlicht mapping. Indian Journal Pure Applied Mathematics, 8, 864-872.

Deniz, E. (2013). Certain subclasses of bi-univalent functions satisfying subordinate conditions. J. Class Anal., 2, 49-60.

Duren, M.P. (1983). Univalent Functions, in Grundlehren Math. Wissenschaften, Springer-Verlag, Berlin, New York, Heidelberg and Tokyo.

Frasin, B.A. & Aouf, M.K. (2011). New subclasses of bi-univalent functions. Applied Mathematics Letter, 25, 1569-1573.

Lashin, A.Y. (2016). On certain subclasses of analytic and bi-univalent functions. Journal of the Egyptian Mathematical Society, 24, 220-225.

Lewin, M. (1967). On a coefficient problem for bi-univalent functions. Proceedings of the American Mathematical Society, 18, 63-68.

Ma, W.C. & Minda, D. (1994). A unified treatment of some special classes of univalent functions. Proceedings of the Conference on Complex Analysis, Tianjin, 1992, Conf. Proc. Lecture Notes Anal., vol. I, pp. 157-169, International Press, Cambridge.

Magesh, N. & Yamini, J. (2013). Coefficient bounds for certain subclasses of bi-univalent functions. Int. Math. Forum,8, 1337-1344.

Murugusundaramoorthy, G., Magesh, N. & Prameela, V. (2013). Coefficient bounds for certain subclasses of bi-univalent function. Abstract and Applied Analysis, 3 pages.

Netanyahu, E. (1969). The minimal distance of the image boundary from the origin and the second coefficient of a univalent functions in $|z| < 1$. Archive for Rational Mechanics and Analysis,32 (2), 100-112.

Porwal, S. & Darus, M. (2013). On a new subclass of bi-univalent functions. J. Egypt. Math. Soc.,21 (3), 190-193.

Robertson, M.S. (1961). Applications of the subordination principle to univalent functions. Pacific Journal of Mathematics, 11, 315-324.

Sakaguchi, K. (1959). On a certain univalent mapping. Journal of the Mathematical Society of Japan, 11, 72-75.

Srivastava, H.M., Bulut, S., Caglar, M.&Yagmur, N. (2013). Coefficient estimates for a general subclass of analytic and bi-univalent functions. Filomat, 27 (5), 831-842.

Srivastava, H.M., Mishra, A.K. & Gochhayat, P. (2010). Certain subclasses of analytic and bi-univalent functions. Applied Mathematics Letter, 23, 1188-1192.

Xu, Q.H., Gui, Y.C. & Srivastava, H.M. (2012). Coefficient estimates for a certain subclass of analytic and bi-univalent functions. Applied Mathematics Letters, 25, 990-994.

Zaprawa, P. (2014). Estimates of Initial Coefficients for Bi-Univalent Functions. Abstract and Applied Analysis,2014, Article ID 357480, 6 pages.