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Recent Developments in Mixed Poisson Distributions

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Mixed Poisson distributions are a class of distributions arising from the Poisson mean fluctuating as a random variable. Mixed Poisson distributions have been applied in diverse disciplines for modelling non-homogeneity in populations. This paper brings together recent work on this class of distributions with focus on specific models, computation and simulation, applications to stochastic and data modelling.

Keywords: count distributions; distribution theory; mixtures; mixing distributions; over dispersion; Stochastic modelling

I. INTRODUCTION

The Poisson distribution is a basic model for the analysis of count data but is restricted in its applications due to the equality of its mean and variance (equi-dispersion). A popular alternative is the class of mixed Poisson distributions when the count frequency data exhibit overdispersion (variance greater than the mean). Overdispersion is common in a count data. For example, in accident proneness modelling, the probability distribution of accidents for a group of individuals is modelled by the negative binomial (NB) distribution, a mixed Poisson distribution. Some of the earliest work on mixed Poisson distributions was dated as far back as a century ago, such as the paper on accident proneness by Greenwood and Yule (1920). A comprehensive survey of mixed Poisson distributions was given by Karlis and Xekalaki (2005). Gupta and Ong (2005) reviewed their applications in fitting very long-tailed data. The book on univariate discrete distributions by Johnson *et al.* (2005) gave a useful summary of properties of mixed Poisson distributions. Since the review paper of Karlis and Xekalaki in 2005, there is still much interest and applications of mixed Poisson distributions; see, for instance, Iyer-Biswas and Jayaprakash (2014) and Simeunović *et al.* (2018). Due to this, the objective of this paper is to bring together recent works on mixed Poisson distributions. We first define the class of mixed Poisson distributions. Let X be a Poisson random variable with parameter λ and probability mass function (pmf):

$$Pr(X=k|\lambda)=e^{-\lambda}\frac{\lambda^k}{k!}, k=0,1,2,3,\ldots,\lambda>0 \tag{1}$$

If the Poisson mean λvaries as a random variable $\Lambda$with probability density function (pdf) $g(\lambda)$, a mixed Poisson distribution is obtained. It has pmf given by:

$$Pr(X=k)=\int_0^\infty e^{-\lambda}\frac{\lambda^k}{k!}g(\lambda)d\lambda. \tag{2}$$

The pdf $g(\lambda)$ is known as the mixing distribution. In the accident proneness model mentioned above, the mixing distribution has the gamma pdf which results in the NB distribution (Greenwood and Yule, 1920). Sometimes the mixed Poisson distribution is considered with a scale parameter θ:

$$Pr(X=k)=\theta^k\int_0^\infty e^{-\theta\lambda}\frac{\lambda^k}{k!}g(\lambda)d\lambda, \theta>0\ . \tag{3}$$

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In actuarial science, θ is known as the exposure-to-risk. In general, if $f(x|\lambda)$ is a pdf indexed by a parameter λ that varies as a random variable $\Lambda$with pdf $g(\lambda)$, the pdf:

$$f(x) = \int_0^\infty f(x|\lambda)g(\lambda)d\lambda \quad (4)$$

is known as the pdf of a mixture distribution. Note that the mixing distribution may be continuous, discrete or distribution with finite steps. For the finite-step distribution, the mixture distribution is known as a finite mixture distribution. Comprehensive reviews and compilation of properties and applications of finite mixture distributions are found in a number of textbooks, for example, McLachlan and Peel (2000). For other references, refer to Karlis and Xekalaki (2005). Several international conferences and special journal issues on finite mixtures showed the continuing interest in this class of models. As an example, see 4th Special Issue on Advances in mixture models, Computational Statistics and Data Analysis, Volume 132, April 2019.

Mixture models have been applied in diverse areas such as robust statistics, classification, Bayesian statistics, computer generation of random samples, latent structure models, and many others. In cluster analysis (classification), a finite mixture model is a natural representation of the components in the population. Ong (1992) considered mixture models to facilitate the generation of bivariate binomial samples with given marginal distributions and correlation.

Section II gives a review of some basic properties and results of the mixed Poisson distributions. In Section III, new mixed Poisson models are discussed. Sections IV and V consider simulation and computation of mixed Poisson distributions and applications to construct new stochastic models and data analysis. This complements the works mentioned above. Finally, in Section VI we briefly conclude.

II. SOME PROPERTIES OF MIXED POISSON DISTRIBUTIONS

Karlis and Xekalaki (2005) have summarized the important and interesting properties of the class of mixed Poisson distributions. Nikoloulopoulos and Karlis (2008) presented a comparison of NB, Poisson-inverse Gaussian and generalized Poisson distributions from aspects such as the tail length and model fit for varying degrees of over dispersion. Recently, Kuba and Panholzer (2016) reviewed the properties of mixed Poisson distributions and probabilistic aspects of the Stirling transform and presented a new simple limit theorem using expansions of factorial moments. The authors also presented unifying and extension of earlier results on the applications of mixed Poisson distributions in the analysis of random discrete structures, and several new results on triangular urn models.

A few basic properties and important results will be given in this section. A number of properties of the mixed Poisson distribution (2) or (3) are inherited from the mixing distribution with pdf $g(\lambda)$. Some of these properties are stated here without proof.

Property 1: Let $G(z)$ be the probability generating function of the mixed Poisson distribution (3) and $M_\Lambda(t)$ be the moment generating function of the mixing distribution. Then $G(z)$ is given by:

$$G(z) = M_\Lambda\big(\theta(z-1)\big) \quad (5)$$

Property 2 (Holgate, 1970): The mixed Poisson distribution is unimodal if the absolutely continuous mixing pdf $g(\lambda)$ is unimodal.

A random variable X is said to have an infinitely divisible distribution if its characteristic function $\phi(t)$can be written as $[\varphi_n(t)]^n$, where $\varphi_n(t)$ are characteristic functions for any $n \geq 1$.

Property 3 (Maceda, 1948): The mixed Poisson distribution is infinitely divisible if the mixing distribution is infinitely divisible.

Infinite divisibility is connected to the important class of Poisson-stopped sum (compound or generalized Poisson) distributions. A discrete distribution is said to be a Poisson-stopped sum distribution if it can be represented as the distribution of a random sum of N independent and identically distributed random variables where N is a Poisson random variable.

Property 4 (Feller, 1968): An infinitely divisible discrete distribution is a Poisson-stopped sum distribution.

In general, every infinitely divisible probability distribution is a limit of Poisson-stopped sum Poisson distributions (Lukacs, 1970). Property 3 together with Property 4 link the mixed Poisson distributions to the Poisson-stopped sum distributions.

Allowing the Poisson mean to vary as a random variable alters the properties and characteristics of the Poisson distribution. Shaked's (1980) Two Crossings Theorem showed that the mixed Poisson distribution has a larger probability of zeros and a longer tail than a Poisson distribution with the same mean.

Two Crossings Theorem (Shaked, 1980) Let X be mixed Poisson distributed with mean μ. There are two integers $0 \leq x < y$, such that:

$$\begin{gathered} Pr(X=k) \geq e^{-\mu}\frac{\mu^k}{k!}, k=0,1,2,..,x \\ Pr(X=k) \leq e^{-\mu}\frac{\mu^k}{k!}, k=x,1,2,..,y \\ Pr(X=k) \geq e^{-\mu}\frac{\mu^k}{k!}, k \geq y+1 \end{gathered} \tag{6}$$

Although different mixed Poisson distributions are obtained by a different choice of the mixing distributions, this theorem shows that to model particular features of the data such as very high zero counts, extra-long tail or large overdispersion, a proper selection of the mixing distribution is required.

III. SPECIFIC MIXED POISSON DISTRIBUTIONS

There is interest in proposing and studying specific mixed Poisson models, especially for empirical modelling. A particular distribution that has attracted the attention of researchers is the Poisson-Lindley distribution. The Poisson-Lindley distribution and proposed generalizations have simple closed forms for the pmf. Sankaran (1970) proposed a single-parameter Poisson-Lindley distribution which has a closed-form pmf given by:

$$P(X=x) = \frac{\theta^2(\theta+2+x)}{(\theta+1)^{x+3}}, x=0,1,2,3,\ldots,\theta>0. \tag{7}$$

The mixing distribution is the Lindley distribution with pdf (Lindley, 1958):

$$f(\lambda) = \frac{\theta^2}{1+\theta}(1+\lambda)e^{-\theta\lambda}, \lambda>0, \theta>0. \tag{8}$$

The single-parameter Poisson-Lindley distribution may be written as a finite mixture of a geometric $(\theta/(1+\theta))$ distribution and a NB $(2,\theta/(1+\theta))$ distribution with mixing proportion $\theta/(1+\theta)$. Gómez-Déniz *et al.* (2012) proposed multivariate extensions of this Poisson-Lindley distribution with a flexible covariance structure and discussed its properties such as estimation methods. This multivariate Poisson-Lindley application was illustrated in the modelling of automobile insurance claim counts for computation of bonus-malus premiums.

Ghitany and Al-Mutairi (2009) investigated the properties of the method of moments and maximum likelihood estimators of the Poisson-Lindley distribution's single parameter, such as its biasness and asymptotic properties. It is found that both estimators are almost equally efficient. Mahmoudi and Zakerzadeh (2010) generalized the Poisson-Lindley distribution to a two-parameter generalized Poisson-Lindley distribution with pmf:

$$\begin{gathered} P(X=x) = \frac{\Gamma(\alpha+x)}{x!\,\Gamma(\alpha+1)}\frac{\theta^{\alpha+1}}{(\theta+1)^{x+\alpha+1}}\left(\alpha+\frac{\alpha+x}{\theta+1}\right), \\ x=0,1,2,3,\ldots,\alpha,\theta>0 \end{gathered} \tag{9}$$

The mixing distribution for this generalized Poisson-Lindley distribution (Zakerzadeh & Dolati, 2009) has pdf:

$$f(\lambda) = \frac{\theta^{\alpha+1}}{1+\theta}\frac{\lambda^{\alpha-1}}{\Gamma(\alpha+1)}(\theta+\lambda)e^{-\theta\lambda}, \lambda>0, \alpha,\theta>0. \tag{10}$$

Properties of the distribution and performance of various parameter estimation methods were studied. Simulation of variables from the distribution was also provided.

Bhati *et al.* (2015) proposed another new generalized Poisson-Lindley distribution when the mixing distribution follows a two-parameter Lindley distribution (Shanker *et al.*, 2013) defined as:

$$f(\lambda) = \frac{\theta^2}{\alpha+\theta}(1+\alpha\lambda)e^{-\theta\lambda}, \lambda>0, \alpha,\theta>0. \tag{11}$$

The resulting pmf is:

$$\begin{gathered} P(X=x) = \frac{\theta^2}{(\theta+\alpha)(\theta+1)^{x+1}}\left(1+\frac{\alpha(1+x)}{\theta+1}\right), \\ x=0,1,2,3,\ldots \end{gathered} \tag{12}$$

Basic properties have been derived. The pmf still retains the finite mixture interpretation of geometric and negative distributions but with mixing proportion $\theta/(\alpha+\theta)$. That is, an extra parameter α is introduced through the mixing proportion. Bhati *et al.* (2015) gave an actuarial application to collective risk model by considering the proposed distribution as primary distribution and exponential and Erlang as secondary distributions. It is shown that the model performed better than competing models for some real data sets.

Recently, Das *et al.* (2018) proposed a further generalization of the Poisson-Lindley distribution known as the three-parameter Poisson-Lindley distribution. The mixing distribution used has pdf:

$$f(\lambda)=\frac{\theta^2}{\beta+\theta\alpha}(\alpha+\beta\lambda)e^{-\theta\lambda}, \lambda>0, \beta>0, \theta>0. \quad (13)$$

The three-parameter Poisson-Lindley distribution has pmf:

$$P(X=x)=\frac{\theta^2}{(\theta+1)^{x+2}}\left(1+\frac{\alpha+\beta x}{\theta\alpha+\beta}\right),$$
$$x=0,1,2,3,\ldots \quad (14)$$

Statistical properties of this new distribution were investigated, and it was found to be a viable alternative to the Poisson-Lindley and two-parameter Poisson-Lindley distributions.

Sarabia and Gomez-Deniz (2011) extended the results from Holla and Bhatacharya (1965) to derive two multivariate versions of the mixed Poisson-beta distribution, one of which is based on the Sarmonov-Lee model (Lee, 1996). They examined the estimation methods and discussed examples of application in accident analysis and modelling of fault counts in lenses.

Using the weighted exponential distribution as the mixing distribution, Zamani *et al.* (2014) introduced a two-parameter Poisson-weighted exponential distribution and regression model and subsequently introduced a bivariate extension of the distribution.

Bhati *et al.* (2017) studied a new mixed Poisson distribution where the transmuted exponential distribution is the mixing distribution. Distributional properties and parameter estimation were considered for this mixed Poisson distribution and illustrate with an actuarial application in the context of aggregate claim distribution.

Low *et al.* (2017) proposed a generalized Sichel distribution obtained by using the extended generalized inverse Gaussian distribution as mixing distribution. This mixed Poisson distribution is introduced to model simultaneously overdispersion, high zero-inflation and excessive heavy-tails in count data sets. The generalized Sichel distribution has pmf given by:

$$P(X=k)=$$
$$\frac{1}{(2/\delta)(b/a)^{\frac{\lambda}{2\delta}}K_{\lambda/\delta}(2\sqrt{ab})}\int_0^\infty \frac{e^{-\theta}\theta^k}{k!}\theta^{\lambda-1}\, exp(-a\theta^\delta-b\theta^{-\delta})\, d\theta \quad (15)$$

which can be written as:

$$P(X=k)=$$
$$\left(\frac{1}{K_{\lambda/\delta}(2\sqrt{ab})k!}\right)\sum_{j=0}^\infty \frac{(-1)^j}{j!}\left(\frac{b}{a}\right)^{(j+k)/2\delta} K_{(j+k+\lambda)/\delta}(2\sqrt{ab}) \quad (16)$$

where $K_\nu(z)$ is the modified Bessel function of the third kind with index ν.

Gómez-Déniz and Calderin-Ojeda (2018) studied the properties of the mixed Poisson distribution which was obtained by considering the reciprocal inverse Gaussian distribution as the mixing distribution. They also considered parameter estimation via an EM-type algorithm in its regression model. Application of this Poisson-reciprocal inverse Gaussian distribution on modelling claim frequency is found to be competitive with the more popular NB and Poisson-inverse Gaussian distribution.

Habibi and Asgharzadeh (2018) constructed a new mixed Poisson distribution by mixing the Poisson distribution with the binomial, exponential 2 distribution. Basic properties, parameter estimation by method of moments and maximum likelihood and applications to real data sets were considered.

Recently, Ong *et al.* (2019) examined probabilistic properties of the non-central negative binomial distribution (NNBD) which is both a mixed Poisson and a Poisson stopped-sum (Ong & Toh, 2001) or compound distribution. These properties are log-concavity, discrete self-decomposability, unimodality and asymptotic behaviour. The construction as a mixed Poisson process is also considered. The non-central negative binomial distribution has applications in photon and neural counting, statistical optics, astronomy and a stochastic reversible counter system. The NNBD is a mixed Poisson distribution with the non-central gamma distribution as mixing distribution. The non-central gamma pdf is given by:

$$g(\theta) = a^{(v+1)/2}(\theta/\lambda)^{(v-1)/2}\exp\{-(\lambda + a\theta)\}\, I_{(v-1)}\left(2\sqrt{a\lambda\theta}\right),$$
$$\theta > 0, a, v, \lambda > 0. \tag{17}$$

The pmf is given in terms of Laguerre polynomials $L_k^{(\alpha)}(x)$ orthogonal over (0,∞) with respect to $x^{\alpha-1}e^{-x}$:

$$\Pr(k) = e^{-\lambda p}p^k q^v L_k^{(v-1)}(-\lambda q) \tag{18}$$

The Laguerre polynomials $L_k^{(\alpha)}(x)$ are defined by:

$$L_k^{(\alpha)}(x) = \frac{(\alpha+1)_k}{k!}\ {}_1F_1(-k, \alpha+1; x) \tag{19}$$

where ${}_1F_1(a, b; x)$ is the confluent hypergeometric function given by:

$${}_1F_1(a; b; z) = \sum_{i=0}^{\infty} \frac{(a)_i}{(b)_i}\frac{z^i}{i!}, (a)_i = \frac{\Gamma(a+i)}{\Gamma(a)}. \tag{20}$$

Apart from the bivariate generalizations of the NNBD cited in Ong *et al.* (2019), another generalization was given recently by Ong and Ng (2013).

Kempton (1975) obtained a generalization of NB distribution as a mixed Poisson distribution with mixing distribution having pdf:

$$g(\lambda) = \frac{b^p\lambda^{p-1}}{B(p,q)(1+b\lambda)^{p+q}}. \tag{21}$$

The generalized NB distribution has pmf:

$$P(k) = \frac{1}{B(p,q)}\int_0^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}\ \frac{b^p\lambda^{p-1}}{(1+b\lambda)^{p+q}}d\lambda,$$
$$= \frac{\Gamma(p+k)}{k!\,B(p,q)b^q}\ \psi\left(p+q,\ q+1-k; \frac{1}{b}\right),$$
$$k = 0,1,2,\ldots, \tag{22}$$

where ψ is the confluent hypergeometric function of the second kind (Erdelyi, 1953). Ong (1995) and Ong and Muthaloo (1995) have considered computation and statistical properties of this generalized NB and another mixed Poisson with the inverted beta as mixing distribution. Ong and Low (2019) examined the properties, applications to empirical modelling, and computation of Kempton's generalized NB distribution probabilities.

In the Bayesian framework for mixed Poisson distributions, the mixing variable ω follows the distribution $G(\omega; \boldsymbol{\eta})$, and the count variable X follows a Poisson distribution with mean $\lambda = \mu\omega$, where $\mu > 0$ and $\boldsymbol{\eta}$ is a vector of parameters that characterizes G. The marginal pmf of X is thus given as:

$$f(x; \mu, \boldsymbol{\eta}) = \int_0^{\infty} f_P(x; \mu\omega)g(\omega; \boldsymbol{\eta})d\omega, \tag{23}$$

where $g(\omega; \boldsymbol{\eta})$ denotes the pdf of ω and $f_P(x; \mu\omega)$ denotes the pmf of Poisson distribution with mean $\mu\omega$. Hassanzadeh and Kazemi (2016) introduced the log-skew-normal as a mixing prior resulting in the Poisson-log-skew-normal distribution and presented its main properties. This new distribution can account for both overdispersion and zero-inflation. In this case, evaluation of the likelihood functions is performed via numerical methods and the proposed distribution is found to perform very well in the presence of overdispersion and zero-inflation.

In regression modelling, Barreto-Souza and Simas (2016) obtained a general class of mixed Poisson regression models, including the NB and Poisson-inverse Gaussian models. The regression structure and EM-based parameter estimation for the mean and dispersion parameters with diagnostic measures were considered with illustration on a dataset on the number of absences in two urban high schools.

Gómez-Déniz and Calderin-Ojeda (2016) proposed the exponential-inverse Gaussian distribution of Bhattacharya and Kumar (1986) to be the mixing distribution resulting in a mixed Poisson regression model with closed-form expressions for its factorial moments. The regression model was found to fit well on a well-known healthcare demand dataset.

Gómez-Déniz *et al.* (2016) considered a Poisson-mixed inverse Gaussian distribution and regression model whereby a two-component mixture of the inverse Gaussian and the length-biased inverse Gaussian distribution was applied as the mixing distribution. Parameter estimation of the model via maximum likelihood estimation was studied. The application of the regression model was illustrated on a dataset on the number of hospitals stays among the elderly population.

IV. COMPUTATION AND SIMULATION

Even though the mixed Poisson distributions form an important class of distributions in applications, the complicated probability distributions of many of these distributions hampered their applications. Some recent work to aid computation and simulation for statistical inferences are presented below.

Ghitany *et al.* (2012) proposed a general EM algorithm for maximum likelihood estimation of a class of multivariate mixed Poisson regression models with special focus on the multivariate NB, Poisson-inverse Gaussian and Poisson-lognormal regression models. The authors analyzed the demand for health care in Australia dataset (Cameron & Trivedi, 1998). The univariate situation has been examined by Karlis (2001, 2005).

Ong and Lee (2008) gave an envelope rejection method for generating NB random samples by exploiting the mixed Poisson formulation for the NB distribution. The envelope rejection is based on a simple probability distribution inequality derived from the rejection result of Tadikamalla (1978) for the gamma distribution.

Izsák (2008) examined the computation of probabilities and maximum likelihood estimation for the Poisson-lognormal distribution by providing a sharp approximation of the Poisson-lognormal integrals probabilities and illustrated its application in modelling species abundance data.

Chatelain *et al.* (2009) presented a maximum pairwise likelihood approach for parameter estimation of multivariate mixed Poisson distributions with multivariate Gamma distribution as the mixing distribution. They investigated the effectiveness of this approach and applied it in the change detection problem in image processing.

Extending the generalized linear mixed model to correlated count response data, Weems and Smith (2018) investigated the robustness of the maximum likelihood estimators of the Poisson-inverse Gaussian model when the distribution of its random effects is misspecified.

Ong *et al.* (2019) derived a general technique for computation of mixed Poisson probabilities by Monte Carlo sampling. This general method applies to any mixed Poisson distribution with arbitrary mixing distribution. Computational speed and accuracy of this technique is exemplified with the Poisson-inverse Gaussian distribution as a benchmark since the probabilities of this distribution can be computed with good accuracy. The proposed method is also applied to compute Poisson-lognormal probabilities, a popular species abundance model. This computation method of mixed Poisson probabilities applied in the Expectation-Maximization (EM) algorithm for maximum likelihood estimation of Poisson-lognormal parameters is shown to avoid numerical problems encountered by existing techniques.

There is an R package MixedPoisson for parameter estimation of different popular mixed Poisson models using the EM algorithm for practitioners.

V. APPLICATIONS TO STOCHASTIC AND DATA MODELLING

Mixed Poisson distributions, in particular the NB distribution, have found applications in constructing integer time series models and in statistical data analysis in a variety of settings. To cater to the integer nature of the time series, for example, an integer autoregressive (INAR(1)) time series $\{X_t\}$ is defined as:

$$X_t = \alpha \circ X_{t-1} + \varepsilon_t \tag{24}$$

where $\alpha \in [0,1]$ and ε_t is the innovation term. '$\circ$' is the thinning operator and is defined as:

$$\alpha \circ X_{t-1} = \sum_{i=1}^{X_{t-1}} B_i \tag{25}$$

B_i is the Bernoulli random variables with the success probability of α. In other words, the thinning operation denotes a stopped sum of Bernoulli random variables.

Zhu (2011) examined a negative binomial GARCH model and gave stationarity conditions and the autocorrelation function. Three approaches for estimation were presented with the focus on the maximum likelihood estimation.

Christou and Fokianos (2014) studied inference and diagnostics for count time series on regression models, particularly NB processes that include a feedback mechanism. For this class of processes, probabilistic properties and quasi-likelihood estimation were considered, and it is shown that the resulting estimators are consistent and asymptotically normally distributed.

Barreto-Souza (2019) considered mixed Poisson INAR(1) processes. The proposed INAR(1) processes have marginal mixed Poisson distributions. A condition for this class of INAR processes to be well-defined has been established, and some statistical properties are studied. Estimators for the parameters are proposed with proofs of their consistency and asymptotic normality. The finite-sample performance of the proposed estimators is evaluated by Monte Carlo simulation and application to a real data set exemplified the utility of the proposed models.

In an application in healthcare, Henríquez and Castrillón (2011) showed that the mixed Poisson distribution works well in computing tumour control probability which is defined as the probability of destroying every clonogen in a tumour as a result of a radiation therapy treatment. They found that the mixed Poisson distribution is able to account for the inhomogeneity in the absorbed dose throughout the tumour volume, thus providing a more flexible method. In this case, pdf of the mixing distribution is derived as $g(\lambda) = \frac{1}{\alpha\lambda} DVH_d\left(-\frac{\log\lambda}{\alpha}\right)$, where DVH is the dose-volume histogram that describes the probability distribution of absorbed dose along with the tumour.

By using mixed Poisson distributions, Iyer-Biswas and Jayaprakash (2014) considered the interplay between stochastic gene expression and system design using simple stochastic models of auto-activation and auto-inhibition.

In accident analysis, the NB distribution is one of the commonly used models for predicting motor vehicle crashes. Lord *et al.* (2005) compared the NB distribution with other commonly used statistical models in the accident analysis literature and recommended guidelines for choosing appropriate models when investigating motor vehicle crash data. In this context, mixed Poisson distributions such as the NB distribution has a natural interpretation in accounting for unobserved heterogeneity in the response variable. Subsequently, Lord and Miranda-Moreno (2008) investigated the effects of low sample means and small sample size on the maximum likelihood estimation of the dispersion parameter in Bayesian modelling of motor vehicle crashes using the NB and Poisson-lognormal models. Cheng *et al.* (2013) applied the Poisson-Weibull distribution in the regression context to analyze motor vehicle crash data. This mixed Poisson distribution has the Weibull distribution as mixing distribution. The pmf of the Poisson-Weibull model does not have a closed-form and is an impediment in applications. This is resolved with the Bayesian interpretation of the mixed Poisson formulation so that the Markov chain Monte Carlo (MCMC) technique can be employed in the parameter estimation and inference. It is remarked here that the Monte Carlo computation of mixed Poisson probabilities proposed by Ong *et al.* (2019) may be used.

The Poisson-Lindley distribution of Sankaran (1970) was considered by Hernández-Bastida *et al.* (2011) as the primary distribution for modelling the number of claims in a collective risk model to calculate the Bayes premium in actuarial science as well as to determine regulatory capital in operational risk analysis. Simeunović *et al.* (2018) examined the process of determining premium rates in automobile liability insurance using the bonus-malus system. A bonus - malus system has been constructed based on mixed Poisson models, where the expected value principle is used to determine the net premium.

VI. CONCLUDING REMARKS

This review has gathered some recent work on mixed Poisson distributions from the following areas: new univariate and multivariate mixed Poisson models or further study of existing models, applications in stochastic modelling and data analysis, simulation and computation. Mathematically it is straightforward to derive new mixed Poisson models by choosing the mixing distribution. The Two Crossings Theorem shows that relative to the Poisson distribution, the mixing distribution alters the zero counts and tail length. Hence for statistical data analysis, a judicious choice of mixing distribution is necessary to cater for high zero counts or long tail length in data. Mixed

Poisson distributions may have intractable probability functions even though the mixing distribution such as the lognormal or Weibull distribution has a simple form. A number of the new mixed Poisson distributions have tractable forms like the generalized Poisson-Lindley distribution. The general Monte Carlo simulation approach may be applied to compute complicated mixed Poisson probabilities and maximum likelihood by EM algorithm for mixed Poisson distributions as exemplified by the Poisson-lognormal distribution.

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