ASM Sc. J., 14, Special Issue 1, 2021 for ICSTSS2018, 25-31

The Radimacher – Menchoff Theorem for Eigenfunction Expansions of Elliptic Differential Operators and its Application for Almost Everywhere Convergence of the Double Fourier Series Summed over Elliptic Level Curves

Anvarjon Ahmedov[1*], Ehab Materneh[2] and Muzamir Fatimah[3]

[1]*Centre for Mathematical Sciences, Universiti Malaysia Pahang, Malaysia*

[2]*Institute for Mathematical Research, Universiti Putra Malaysia, Selangor, Malaysia*

[3]*Earth Resources Sustainability Centre (ERSC), Universiti Malaysia Pahang, Malaysia*

The current work is devoted for the problems on the almost everywhere (a.e.) convergence of double Fourier series (DFS). The obtained result is the generalization of the classical Radimacher-Menchoff (R-M) theorem for the eigenfunction expansions of second order elliptic operators.

Keywords: a.e.convergence; DFS; elliptic operators; Radimacher-Menchoff theorem

I. INTRODUCTION

In this work, we investigate problems on the almost everywhere (a.e.) of double Fourier series (DFS) summed over elliptic curves. DFS plays an important role in solving boundary value problems arising in the theory of elliptic differential operators (DOs). The second order elliptic DOs have many applications in quantum mechanics and many areas of the engineering sciences. Such operators are unbound and have self adjoint extension in the Hilbert classes, which is closely connected with the DFS summed over the levels of elliptic polynomials. The main problem of the harmonic analysis is reconstruction of the function from its Fourier expansion. Obtaining the sufficient conditions for the a.e. convergence of the DFS of the functions from different classes gives answer to the main problem of the harmonic analysis in the classes of Liouville. By applying the established new Radimacher-Menchoff (R-M) Theorem for general spectral expansions of elliptic operators, one can prove a.e. convergence of the DFS summed over domains bounded by levels of an elliptic polynomials. We note that DFS summed over the circular curves are connected with the well known Laplace operator and studied by many authors such as Alimov *et al.* (1992), Bastic (1983) and Geok *et al.* (2008). Motivated by the problems on the a.e. convergence of Fourier series (FS), Plancherel (1913) studied the a.e. convergence of orthogonal series. Rademacher (1922) and Menchoff (1926) proved the following theorem.

Theorem (R-M) *Let* $\{\phi_n(x)\}$ *be a system of orthonormal functions in the interval* (a, b). *If the numbers* $\{c_n\}$ *satisfy the condition*

$$\sum_{n=1}^{\infty} |c_n|^2 ln^2(\mathrm{n}) < \infty,$$

then, the FS

$$\sum_{n=1}^{\infty} c_n \phi_n(x)$$

is a.e. convergent in the interval (a, b).

The current research extends the results on the a.e. convergence of the DFS for the elliptic partial sums. Here, we introduce the goals of this research which are: 1. To obtain the sufficient conditions for the a.e. convergence of the DFS

*Corresponding author's e-mail: anvarjon@ump.edu.my

summed by elliptic level curves 2. To generalize the Menchoff-Rademacher Theorem to the spectral decompositions of the self-adjoint DOs.

In order to achieve the target, we first investigate special partial sums summed over elliptic level curves. Next, we generalize the obtained estimation for general maximal operators by deep analysis of the elliptic partial sums. At the final step, we apply methods of harmonic analysis to prove a.e. convergence of the DFS.

$$\mathrm{T}^2 = \{(x, y) \in \mathbb{R}^2 : -\pi \le x < \pi, -\pi \le y < \pi\}.$$

We consider functions, which are 2π -periodic in each variable. A function $f(x, y)$ belongs to the class $L_2(\mathrm{T}^2)$, if the following double integral is finite: $\iint_{\mathrm{T}^2} |f(\xi,\eta)|^2 < \infty$, where we use notation $\mathrm{T}^2 = \{(\xi,\eta) \in \mathbb{R}^2 : -\pi \le \xi < \pi, -\pi \le \eta < \pi\}$. The class $L_2(\mathrm{T}^2)$ is normed space with the norm,

$$\|f\|_{L_2(T^2)} = \left(\iint\limits_{\mathrm{T}^2} |f(\xi,\eta)|^2 \mathrm{d}\xi d\eta\right)^{\frac{1}{2}}$$

We investigate summability problems of the DFS of a function $f(\xi,\eta) \in L_2(\mathrm{T}^2)$:

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \hat{f}_{nm} e^{i(n\xi+m\eta)} \qquad (1)$$

where $\hat{f}_{nm}$ is the Fourier coefficients of the function f:

$$\hat{f}_{\mathrm{nm}} = (2\pi)^{-2} \iint\limits_{\mathrm{T}^2} f(\xi,\eta) e^{-i(n\xi+m\eta)} d\xi d\eta.$$

Recall famous Perceval formula, which establishes connection between the norm of the function and its Fourier coefficients:

$$\iint\limits_{\mathrm{T}^2} |f(\xi,\eta)|^2 d\xi d\eta = (2\pi)^2 \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} |\hat{f}_{nm}|^2 \qquad (2)$$

It is well known that the FS of the function f is convergent in the norm of $L_2(\mathrm{T}^2)$. We note here that the convergence in L_2 − norm does not imply pointwise convergence of the FS of the function from $L_2(\mathrm{T}^2)$. Define a partial sums of the DFS in Eq. (1) summed over the levels of the elliptic polynomial $p(\xi,\eta) = a_{2,0}\xi^2 + 2a_{1,1}\xi\eta + a_{0,2}\eta^2$ by the following

$$E_\lambda f(\xi,\eta) = \sum_{p(n,m)<\lambda} \sum \hat{f}_{nm} e^{i(n\xi+m\eta)} \qquad (3)$$

The partial sums E_λ is spectral decomposition corresponding to the following operator,

$$p\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial y}\right) = -a_{2,0}\frac{\partial^2}{\partial x^2} - 2a_{1,1}\frac{\partial^2}{\partial x\,\partial y} - a_{0,2}\frac{\partial^2}{\partial y^2},$$

which is the second order elliptic operator. There were many researchers (Alimov *et al.*, 1992), which devoted to the problems on the convergence of the partial sums of the DFS summed over circular levels. Such DFS corresponds to the self-adjoint extensions of the Laplace operator on torus. In the current work, we captured the influence of the symbol of elliptic DOs to the conditions for the a.e. convergence of FS.

Theorem 1.1 *Let $f \in L_2(\mathrm{T}^2)$ and its Fourier coefficients satisfy the condition*

$$\textstyle\sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} |\hat{f}_{nm}|^2 ln^2(1 + p(n,m)) < \infty \qquad (4)$$

then, the FS in Eq. (1) of the function f a.e. converges to f in the torus T^2 by summation over the elliptic curves

$$\lim_{\lambda\to\infty} E_\lambda f(\xi,\eta) = f(\xi,\eta) \text{ a.e. in } T^2.$$

The a.e. convergence of the series corresponding to the general orthogonal system of functions under conditions similar to the conditions in Theorem 1.1 is obtained by Menchoff and Radimacher (we refer the readers to (Alimov *et al.*, 1992) and (Zygmund, 1959) for reference). Later in the work (Bastis, 1983) the generalization of the Menchoff-Radimacher theorem for spectral expansions of the elliptic operators are established when the sequence of the spectral expansions is taken over the dense set in $\mathbb{R}$. Here, we introduce new method to extend the Menchoff-Radimacher Theorem for general elliptic operators without any conditions like in Bastis (1983). We note that the Menchoff-Radimacher Theorem can be applied to obtain the results on the a.e.

convergence of the eigenfunction expansions connected with elliptic DOs in torus Ahmedov *et al.* (2018) and Ahmedov (2018). In the work of Geok *et al.* (2008), the positioning method for Location Based Services based on non-Taylor series, which includes DFS, is investigated. The current research can be applied to develop Location Based Services positioning tools which can generate position information in convenient way (Van Diggelen & Brown, 1994).

II. THE ESTIMATION FOR LACUNARY MAXIMAL OPERATOR

In this section, we estimate maximal operator summed over the sets $\Upsilon_k = \{(\mathrm{n},\mathrm{m}): p(n,m) < 2^k\}, k = 0,1,2,\ldots$ Let consider the partial sums of the following special type

$$E_{2^k} f(\xi,\eta) = \sum\sum_{p(n,m)<2^k} \hat{f}(n,m) e^{i(n\xi+m\eta)}.$$

The following Lemma shows that for such partial sums the a.e. convergence has a place for the functions from $L_2(\mathrm{T}^2)$ with the additional conditions for the Fourier coefficients:

Lemma 2.1 *Let Fourier coefficients of the function* $f \in L_2(\mathrm{T}^2)$ *satisfy the condition*

$$\sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} |\hat{f}_{nm}|^2 ln^2(1+p(n,m)) < \infty.$$

Then, the partial sums $E_{2^k} f(\xi,\eta)$ *a.e. converges to* $f(\xi,\eta)$ *in the torus* T^2. *Moreover the maximal operator*

$$E_* f(\xi,\eta) = \sup_k \left| E_{2^k} f(\xi,\eta) \right|$$

satisfies the inequality

$$\|E_* f\|_{L_2(T^2)} = O(1) \left\{ \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \left|\hat{f}_{nm}\right|^2 ln^2\big(1+p(n,m)\big) \right\}^{\frac{1}{2}}$$

Proof. Let $f \in L_2(\mathrm{T}^2)$. By using the Plancherel theorem, we obtain

$$\iint_{T^2} \sum_{k=0}^{\infty} |E_{2^k} f(\xi,\eta) - f(\xi,\eta)|^2 d\xi d\eta =$$

$$\sum_{k=0}^{\infty} \iint_{T^2} \left(E_{2^k} f(\xi,\eta) - f(\xi,\eta)\right) \overline{\left(E_{2^k} f(\xi,\eta) - f(\xi,\eta)\right)} d\xi d\eta$$

$$= \sum_{k=0}^{\infty} \sum\sum_{p(n,m)\geq 2^k} |\hat{f}_{\mathrm{nm}}|^2$$

In the latter expression we proceed first complete the summation over elliptic levels $\{(n,m) \in \mathbb{Z}\times\mathbb{Z}: \; p(n,m) = \nu\}$, then by $\{\nu: \; \nu \geq 2^k\} = \{\nu: \; \log_2\nu \geq k\}$:

$$\sum_{k=0}^{\infty} \sum\sum_{p(n,m)\geq 2^k} \left| \hat{f}_{\mathrm{nm}} \right|^2 = \sum\sum_{p(n,m)\geq 1} \sum_{k=0}^{[\log_2 p(n,m)]} \left| \hat{f}_{nm} \right|^2$$
$$= \sum_{\nu=1}^{\infty} \sum_{k=0}^{[\log_2\nu]} \sum_{p(n,m)=\nu} |\hat{f}_{nm}|^2.$$

Using the identity $\sum_{k=0}^{[\log_2\nu]} 1 = 1 + [\log_2\nu]$, we write a latter as follows

$$\sum_{\nu=1}^{\infty} \sum_{k=0}^{[\log_2\nu]} \sum_{p(n,m)=\nu} |\hat{f}_{nm}|^2 =$$

$$= O(1) \sum_{\nu=1}^{\infty} (1+\log_2\nu) \sum_{p(n,m)=\nu} |\hat{f}_{\mathrm{nm}}|^2$$

Then, we finalize the estimation

$$\iint_{T^2} \sum_{k=0}^{\infty} \left|E_{2^k} f - f\right|^2 =$$

$$= O(1) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |\hat{f}_{nm}|^2 \ln\big(1+p(n,m)\big) \quad (5)$$

By virtue of the condition in Eq. (4), we conclude that

$$\iint_{T^2} \sum_{k=0}^{\infty} |E_{2^k} f - f|^2 < \infty.$$

The application of the Theorem of Bepo Levy (Jost, 2005) gives that series

$$\sum_{k=0}^{\infty} |E_{2^k} f - f|^2$$

is a.e. convergent in the torus T^2. The necessary condition for the convergence of series is that the k^{th} term should converge to zero:

$$\lim_{k\to\infty} E_{2^k} f = f \text{ a.e. in } T^2.$$

Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we estimate

$$|E_{2^k} f|^2 = |(E_{2^k} f - f) + f|^2 \leq 2(|E_{2^k} f - f|^2 + |f|^2).$$

By taking the supremum, we estimate

$$|E_* f|^2 \leq 2|f|^2 + 2\sum_{k=0}^{\infty} |E_{2^k} f - f|^2.$$

Integration over the T^2 gives

$$\iint_{\mathrm{T}^2} [E_* f]^2 \leq 2 \iint_{\mathrm{T}^2} |f|^2 + 2 \iint_{\mathrm{T}^2} \sum_{k=0}^{\infty} |E_{2^k} f - f|^2.$$

Finally, by using Eq. (5) in combination of

$$\iint_{T^2} |f|^2 \leq \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |\hat{f}_{\mathrm{nm}}|^2 \ln(1 + p(n,m)),$$

we obtain

$$\iint_{T^2} [E_* f]^2 = O(1) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |\hat{f}_{\mathrm{nm}}|^2 \ln(1 + p(n,m)),$$

which proves the statements of Lemma 2.1.

Lemma 2.2 *The partial summation operator E_λ corresponding to the DFS of the function $f \in L_2(\mathrm{T}^2)$ satisfy the following inequality*

$$\iint_{\mathrm{T}^2} [\sup_{0\leq k<2^\gamma} |E_k f|]^2 \leq \gamma^2 \sum_{p(n,m)<2^\gamma - 1} |\hat{f}_{\mathrm{nm}}|^2.$$

Proof. Let $f \in L_2(\mathrm{T}^2)$. We note that for any complex numbers $z_k,\ k = 0,1,2,\cdots,2^\gamma - 1$ and for any natural number l: $1 \leq l < 2^\gamma$ we have

$$\left| z_l - z_0 \right|^2 \leq \gamma \sum_{j=0}^{\gamma-1} \sum_{i=1}^{2^{\gamma-j}-1} \left| z_{i2^j} - z_{(i-1)2^j} \right|^2$$

Let denote $z_k = E_k f(x,y),\ k = 1,2,3,\cdots,2^\gamma - 1$. We put $z_0 = 0$.

We obtain

$$\iint_{\mathrm{T}^2} [\sup_{0\leq k<2^\gamma} |E_k f|]^2 \leq$$

$$\leq \gamma \sum_{j=0}^{\gamma-1} \sum_{i=1}^{2^{\gamma-j}-1} \iint_{\mathrm{T}^2} \left| E_{i2^j} f - E_{(i-1)2^j} f \right|^2$$

$$\leq \gamma \sum_{j=0}^{\gamma-1} \sum_{i=1}^{2^{\gamma-j}-1} \sum_{(i-1)2^j \leq \mathrm{p}(n,m) \leq i2^j} |\hat{f}_{\mathrm{nm}}|^2$$

$$\leq \gamma \sum_{j=0}^{\gamma-1} \sum_{\mathrm{p}(n,m)<2^\gamma-1} |\hat{f}_{\mathrm{nm}}|^2.$$

We finalize as follows:

$$\iint_{\mathrm{T}^2} [\sup_{0\leq k<2^\gamma} |E_k f|]^2 \leq \gamma^2 \sum_{\mathrm{p}(n,m)<2^\gamma-1} |\hat{f}_{\mathrm{nm}}|^2.$$

Lemma 2.3 *If Fourier coefficients of the function $f \in L_2(T^2)$ satisfy the condition*

$$\sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} |\hat{f}_{\mathrm{nm}}|^2 \ln^2(1 + \mathrm{p}(n,m)) < \infty,$$

Then, for spectral decompositions E_{2^γ} corresponding to the DFS of function f following estimation holds

$$\iint_{T^2} \left[\sum_{\gamma=2}^{\infty} |E_{2^\gamma} f - E_{2^{\gamma-1}} f|\right]^2 = O(1) \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} |\hat{f}_{\mathrm{nm}}|^2 \ln^2(1 + p(n,m)).$$

Proof. Let $f \in L_2(T^2)$ and its Fourier coefficients satisfy the condition

$$\sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} |\hat{f}_{\mathrm{nm}}|^2 \ln^2(1 + p(n,m)) < \infty.$$

By applying the Cauchy-Swartz inequality to the summation

$$\sum_{\gamma=2}^{\infty} |E_{2^\gamma} f - E_{2^{\gamma-1}} f|$$

we obtain

$$\sum_{\gamma=2}^{\infty} |E_{2^\gamma} f - E_{2^{\gamma-1}} f| =$$

$\sum_{\gamma=2}^{\infty} \frac{1}{\gamma-1}(\gamma-1)|E_{2^\gamma}f - E_{2^{\gamma-1}}f|$

$$\leq \left(\sum_{\gamma=2}^{\infty} \frac{1}{(\gamma-1)^2}\right)^{\frac{1}{2}} \times$$

$$\times \left(\sum_{\gamma=2}^{\infty} (\gamma-1)^2 |E_{2^\gamma}f - E_{2^{\gamma-1}}f|^2\right)^{\frac{1}{2}}$$

Integrating over the T^2 and from $\sum_{\gamma=2}^{\infty} \frac{1}{(\gamma-1)^2} = \frac{\pi^2}{6}$ we obtain the estimation

$$\iint_{T^2} \left[\sum_{\gamma=2}^{\infty} |E_{2^\gamma}f - E_{2^{\gamma-1}}f|\right]^2 \leq$$

$$\leq \frac{\pi}{\sqrt{6}} \sum_{\gamma=2}^{\infty} (\gamma-1)^2 \iint_{T^2} |E_{2^\gamma}f - E_{2^{\gamma-1}}f|^2$$

Application of the Lemma 2.2 gives

$$\iint_{T^2} |E_{2^\gamma}f - E_{2^{\gamma-1}}f|^2 =$$

$$= O(1) \sum_{2^{\gamma-1} \leq A(n,m) < 2^\gamma} |\hat{f}_{\mathrm{nm}}|^2$$

Thus, we have

$$\iint_{T^2} \left[\sum_{\gamma=2}^{\infty} |E_{2^\gamma}f - E_{2^{\gamma-1}}f|\right]^2 =$$

$$= O(1) \sum_{\gamma=2}^{\infty} (\gamma-1)^2 \sum_{2^{\gamma-1} \leq A(n,m) < 2^\gamma} |\hat{f}_{\mathrm{nm}}|^2$$

$$= O(1) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \ln^2(1+p(n,m))|\hat{f}_{\mathrm{nm}}|^2.$$

which completes the proof of Lemma 2.3.

III. THE A.E. CONVERGENCE OF DFS

This section is devoted for the a.e. convergence of the DFS. We note that in order to prove a.e. convergence, it is sufficient to obtain the following estimation:

$$\iint_{T^2} [E_* f]^2 =$$

$$= O(1) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |\hat{f}_{nm}|^2 \ln^2(1+p(n,m)) \qquad (6)$$

In order to prove estimation (6) we proceed as follows: we choose an arbitrary integer number $k > 1$ which has representation $k = 2^\gamma + l$, where $\gamma \in \mathbb{N}$ and l is an integer such that $0 \leq l < 2^\gamma$. Therefore,

$$E_* f = \sup_{k \geq 1} |E_k f|$$

$$\leq \sup_{\gamma \geq 1} \sup_{0 \leq l < 2^\gamma} |E_{2^\gamma + l}f - E_{2^\gamma}f| + \sup_{\gamma \geq 0} |E_{2^\gamma}f|$$

$$\leq \sum_{\gamma=1}^{\infty} \sup_{0 \leq l < 2^\gamma} |E_{2^\gamma + l}f - E_{2^\gamma}f|^2 + \sup_{\gamma \geq 0} |E_{2^\gamma}f|$$

$$= I_1 + I_2.$$

Let first estimate I_1. Using Lemma 2.2 we have

$$\iint_{T^2} I_1^2 dxdy \leq \sum_{\gamma=1}^{\infty} \iint_{T^2} \left(\sup_{0 \leq l < 2^\gamma} |E_{2^\gamma + l}f(x,y) - E_{2^\gamma}f(x,y)|\right)^2 dxdy$$

$$\leq \sum_{\gamma=1}^{\infty} \gamma^2 \sum_{2^\gamma \leq p(n,m) \leq 2^{\gamma+1}-1} |\hat{f}_{\mathrm{nm}}|^2$$

$$\leq \sum_{k=3}^{\infty} \log_2^2 k \sum_{k-1 \leq p(n,m) < k} |\hat{f}_{nm}|^2,$$

which can be finalized as follows:

$$\iint_{T^2} I_1^2 dxdy \leq \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \log_2^2(1+p(n,m))|\hat{f}_{\mathrm{nm}}|^2.$$

Let estimate I_2. We note that $E_{2^\gamma}f = E_1 f + \sum_{k=1}^{\gamma} [E_{2^k}f - E_{2^{k-1}}f]$. We conclude that

$$I_2 = \sup_{\gamma \geq 0} |E_{2^\gamma}f| \leq |E_1 f| + \sum_{\gamma=1}^{\infty} |E_{2^\gamma}f - E_{2^{\gamma-1}}f|.$$

Applying the inequality $(a+b+c)^2 \leq 3a^2 + 3b^2 + 3c^2$ we obtain

$$I_2^2 \leq 3|E_1 f|^2 + 3|E_2 f - E_1 f|^2 + 3[\sum_{\gamma=2}^{\infty} |E_{2^\gamma}f - E_{2^{\gamma-1}}f|]^2$$

Integration of the I_2^2 over the T^2 and application of the Lemma 2.2 and Lemma 2.3 give

$$\iint_{T^2} |I_2|^2 dxdy = O(1)\left(\sum_{p(n,m)<1} + \sum_{1\le p(n,m)<2}\right)|\hat{f}_{nm}|^2 +$$

$$+O(1)\sum_{\gamma=2}^{\infty}(\gamma-1)^2 \sum_{2^{\gamma-1}\le p(n,m)<2^{\gamma}} |\hat{f}_{\rm nm}|^2$$

$$= O(1)\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \log_2^2(1+p(n,m))|\hat{f}_{\rm nm}|^2.$$

Collecting the estimations for I_1 and I_2

$$\iint_{T^2} [E_* f]^2 {\rm dxdy} \le 2\iint_{T^2} (I_1^2 + I_2^2) dxdy$$

$$= O(1) \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |\hat{f}_{\rm nm}|^2 \log^2(1+p(n,m)).$$

Which proves the estimation in Equation (6). Thus, using the estimation in Eq. (6) for the maximal operator we prove a.e. convergence of the DFS by standard methods of the Fourier analysis. Let $g(\xi,\eta) \in L_2(\mathrm{T}^2)$ and

$$\hat{g}(n,m) = 0, \forall n,m \in \mathbb{Z}: p(n,m) < k.$$

The maximal operator $E_* g(\xi,\eta)$ we denote by $E_*^{(k)} g(\xi,\eta)$. In the classes $L_2(\mathrm{T}^2)$ we obtain

$$\iint_{T^2} [E_*^{(k)} g]^2 \le C \sum\sum_{p(n,m)>k} |\hat{g}\_{\rm nm}|^2 \log^2(1+p(n,m)).$$

Then for any $\varepsilon > 0$

$$\varepsilon^2 mes\left\{(\xi,\eta) \in \mathrm{T}^2 : E_*^{(k)} g(\xi,\eta) > \varepsilon\right\} \le$$

$$\iint_{T^2} [E_*^{(k)} g]^2,$$

which implies that for any positive ε the $mes\left\{(\xi,\eta) \in \mathrm{T}^2 : E_*^{(k)} g(\xi,\eta) > \varepsilon\right\} \to 0$ as $k \to \infty$. Next, let $f \in L_2(\mathrm{T}^2)$ and we denote by $f_k(\xi,\eta), k = 0,1,2,\dots$ a function from $L_2(\mathrm{T}^2)$, such that

$$\hat{f}_{\rm nm} = \hat{f}_k(n,m), \forall n,m \in \mathbb{Z}: p(n,m) < k.$$

Let us denote by $\Lambda f(\xi,\eta)$ the fluctuation of $E_\lambda f(\xi,\eta)$:

$$\Lambda f = |\lim_{\lambda\to\infty} \sup\{E_\lambda f\} - \lim_{\lambda\to\infty} \inf\{E_\lambda f\}|$$

It is easy to see, that

$$\Lambda f \le 2E_* f.$$

which implies the following

$$\Lambda f = \Lambda\{f - f_k + f_k\}$$

$$= |\limsup_{\lambda\to\infty} E_\lambda(f - f_k) - \liminf_{\lambda\to\infty} E_\lambda(f - f_k)|$$

$$\le 2E_*(f - f_k).$$

Then, the fluctuation is estimated as below:

$$\iint_{T^2} |\Lambda f|^2 \le 4\iint_{T^2} |E_*(f - f_k)|^2 \le$$

$$C\sum\sum_{p(n,m)>k} |\widehat{f - f_k}(n,m)|^2 \ln^2(1+p(n,m)),$$

the right hand side of the latter inequality tends to 0 as $k \to \infty$. Therefore, a.e. in $(\xi,\eta) \in \mathrm{T}^2$ the fluctuation $\Lambda f(\xi,\eta) = 0$. So a.e. on T^2 for partial sums of the functions from $L_2(\mathrm{T}^2)$ we have

$$\lim_{\lambda\to\infty} E_\lambda f(\xi,\eta) = f(\xi,\eta).$$

IV. CONCLUSION

In the present research, we have proved a.e. convergence of DFS summed over the elliptic levels. The sufficient conditions for the a.e. convergence problems are obtained by generalization of the well-known R-M Theorem for spectral decomposition of the elliptic DOs. The estimations for the maximal operator of the elliptic partial sums of the FS, which guarantees the a.e. convergence of FS, is obtained by deep analysis of the partial sums in the case of DFS. The extension of the results of current paper for the general elliptic operators with variable coefficients and pseudo-differential Operators will be one of the main directions of future developments in the theory of DFS.

V. ACKNOWLEDGEMENT

The authors thank RMC of UMP for support of the current research under Universiti Research Grant Scheme RDU190369 and RDU1703279.

VI. REFEREENCES

Ahmedov, A, Materneh, E & Zainuddin, H 2018, 'The a.e. convergence of the eigenfunction expansions from Liouville classes corresponding to the elliptic operators', AIP Conference Proceedings, vol. 1974, pp. 020003.

Ahmedov, A 2018, 'The convergence problems of eigenfunction expansions of elliptic DOs', Journal of Physics: Conf. Ser., vol. 974, no. 1, pp. 012072.

Alimov, SA, Ashurov, RR & Pulatov, AK 1992, 'Multiple FS and Fourier integrals', eds VP Khavin & NK Nikol'skil, in Commutative Harmonic Analysis IV, Encyclopedia of Mathematical Sciences, vol. 42, pp. 1-81, Springer, Berlin, Heidelberg.

Bastis AI 1983, 'On some problems of convergence of spectral expansions of the elliptic DOs', PhD thesis, Moscow State University, Russian.

Geok, TK, Reza, AW & Liew, SC 2008, 'Non-Taylor series based positioning method for location based services', Journal of Information and Communication Technology (JICT) vol. 7, pp. 41-55.

Jost, J 2005, 'The convergence theorems of lebesgue integration theory', in Postmodern Analysis, Springer, Berlin, Heidelberg, pp. 205-215.

Menchoff D 1926, ' Sur les s´eries des fonctions orthogonales II', Fundamenta Math., vol. 8, pp. 56-108.

Plancherel M 1913, 'Sur la convergence des s´eries de fonctions orthogonales', Comptes Rendus Acad. Sci. Paris, vol. 157, pp. 539-542.

Rademacher H 1922, 'Einige S¨atze ¨uber Reihen von allgemeinen Orthogonalfunktionen', Math. Annal., vol. 87, pp. 112-138.

Stein, EM & Weiss, G 1971, 'Introduction to Fourier analysis on Euclidean spaces', pp. 245-286, Princeton University Press, Princeton, New Jersey.

Van Diggelen, F & Brown, A 1994, 'Mathematical aspects of GPS RAIM', Proceedings of 1994 IEEE Position, Location and Navigation Symposium-PLANS'94, pp. 733-738, 11-15 April 1994, Las Vegas, NV, USA.

Zygmund, A 1959, 'Trigonometric series' (Vol. I & II), 3 edn, Cambridge University Press, California, US.