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First and Second-Order Energy Eigenvalues of One-Dimensional Quantum Harmonic and Anharmonic Oscillator with Linear, Quadratic, Cubic and Polynomial Perturbation Potential

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This work is aimed at obtaining the energy eigenvalues for one-dimensional quantum harmonic and anharmonic oscillators perturbed by linear, quadratic, cubic and polynomial potentials. To obtain the solutions of the energy eigenvalues, we employed the time-independent perturbation theory to calculate the first and the second-order energy correction, which we used to obtain the complete generalised energy eigenvalues of the quantum harmonic oscillators with linear, quadratic, cubic and polynomial perturbation potential of the same unperturbed Hamiltonian (H).

Keywords: Quantum Harmonic and Anharmonic Oscillator; Perturbation Theory; Eigenvalue; Perturbation potential; Polynomial perturbation potential

I. INTRODUCTION

The quantum harmonic and anharmonic oscillators have important applications in all areas of physics. Specifically, a harmonic oscillator is a model that has an important analogy when describing physical systems (Halil, 2018; Habtamu, 2019). Harmonic oscillator eigenvalue problems can be solved analytically (Mojtaba & Davood, 2002); but in cases where the exact solution of a problem cannot be found, it is more appreciable to use approximation methods such as perturbation theory, variational method or Wentzel, Kramers and Brillouin (WKB). The perturbation theory approach has been adopted in several ways to obtain the energy eigenvalues of the harmonic oscillator (David, 2016; Nouredine *et al*., 2009).

Most quantum mechanical problems are solved by harmonic oscillator analogy with appropriate boundary conditions. Eigenvalue perturbation theory was first used by Lord Rayleigh in acoustics and Schrodinger in his fundamental series introduced the quantum theory in the 20th century (Mbagwu *et al*., 2020). In remembrance of their contributions, the series is called Rayleigh–Schrodinger perturbation theory but the mathematical foundations were only adopted by Rellich a few years ago (Barry, 1991; Rellich, 1937). In recent years, immense contributions have been made to develop mathematical methods for solving eigenvalues and eigenfunctions of the quantum harmonic and anharmonic oscillators such as the SU (2) group method (SGM) (Kunihiro, 1993). However, perturbation theory is a method used in obtaining an approximate solution of an exact harmonic oscillator problem (Sergei, 2006). Thus, numerical and analytical procedures can be used to calculate these perturbed harmonic oscillators with theoretical contributions (Louisell, 1973; Peidaee *et al*., 2007).

The anharmonic oscillator is one of the key models in solving problems in physics (Bender & Wu, 1969; Bhaumik & Dutta, 1975; Hioe *et al.,* 1978). It can be solved using either the analytical method or approximation method (Hsue & Chern, 1984; Chahjilany *et. al*., 1991; Bacus *et al*., 1995). The

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anharmonic quantum oscillator with quartic potential has been solved using a different approach (McRae & Vrscay, 1997; Ivanov, 1998; Ferńandez & Guardiola, 1993). Quantum anharmonic oscillator with sextic, octic, decatic and the generalised nth term of the potentials has equally been solved recently (Sḱala *et. al.*, 1999; Sharma & Fiase, 2000; Speliotopoulos, 2000). A lot of studies have been done on an anharmonic oscillator with only one perturbation term (Jafarpour *et. al.*, 2003; Vinett & Čižek, 1991; Matamala & Maldonado, 2003). The goal of this work is to calculate the first and second-order energy correction and obtained the generalised energy eigenvalues for the quantum harmonic and anharmonic oscillator with linear, quadratic, cubic and polynomial potential by using the perturbation method whose Hamiltonian as in (Ariel & Philippe, 2018) is given by:

$$H_0 = \frac{P^2}{2m} + \frac{1}{2}\, m\omega^2 x^2 \tag{1}$$

where x *is the* position operator, m *is the* mass of the particle, p *is the* linear momentum operator and ω *is the* angular frequency.

II. ONE DIMENSIONAL QUANTUM HARMONIC OSCILLATOR WITH LINEAR PERTURBATION POTENTIAL

Considering a particle with a linear perturbation Potential given as;

$$H' = \lambda_1 x \tag{2}$$

where λ_1 is the perturbation coefficient. Hence, the total Hamiltonian then becomes;

$$H = H_0 + H' \tag{3}$$

Such that;

$$H = \frac{P^2}{2m} + \frac{1}{2}\, m\omega^2 x^2 + \lambda_1 x \tag{4}$$

At this point, we seek to determine the correction of first-order energy and second-order energy by using first order and second order energy shift formulas.

A. First Order Energy Shift

$$E_n{}^1 = \langle \psi_n{}^0 \,|\, H' \,|\, \psi_n{}^0 \rangle \tag{5}$$

where H' is the perturbed Hamiltonian of the system and $\psi_n{}^0$ is the unperturbed wave function of the system. Hence, substituting Equation (2) into Equation (5) that is;

$$E_n{}^1 = \lambda_1 \langle \psi_n{}^0 \,|\, x \,|\, \psi_n{}^0 \rangle \tag{6}$$

If $\psi_n{}^0 = n$ in which n is the n^{th} eigenfunction. Therefore, we have that;

$$E_n{}^1 = \lambda_1 \langle n \,|\, x \,|\, n \rangle \tag{7}$$

For the harmonic oscillator the position function x is expressed in terms of ladder operators coined from Dirac Operator Technique which is;

$$x = \sqrt{\frac{\hbar}{2m\omega}}\, [a^\dagger + a] \tag{8}$$

where $a^\dagger$ is the step-up (creation) operator and a is the step-down (annihilation) operator. Note: If they don't have the same number of step-up or step-down operators the function will not be part of the state because they will annihilate each other and hence alter the state (Mbagwu *et al.,* 2021).

Thus, substituting Equation (8) into Equation (7), we then have;

$$E_n{}^1 = \lambda_1 \sqrt{\frac{\hbar}{2m\omega}} \tag{9}$$

Applying the operator properties that is;

$$a^\dagger \,|\, n \rangle \sqrt{n+1}\, \,|\, n+1 \rangle \tag{10a}$$

$$a \,|\, n \rangle \sqrt{n}\, |\, n-1 \rangle \tag{10b}$$

Substituting Equation (10a) and (10b) into Equation (9) so we have;

$$E_n{}^1 = \lambda_1 \sqrt{\frac{\hbar}{2m\omega}}\, \left(\sqrt{n+1}\, \langle n \,|\, n+1 \rangle + \sqrt{n}\, \langle n \,|\, n-1 \rangle \right) \tag{11}$$

From the orthogonality condition of ket and bra of;

$$\langle m \,|\, n \rangle = \begin{cases} 1 \; if \; m = n \\ 0 \; if \; m \neq n \end{cases}$$

The ket and bra function in Equation (11) are not equal, thus we can generalise that;

$$E_n{}^1 = 0 \tag{12}$$

Therefore Equation (12) is the first-order energy correction to the eigenvalue E_n of a one-dimensional quantum harmonic oscillator with a linear perturbation potential.

B. Second-Order Energy Shift

Here we seek to obtain the second-order energy correction, using the eigenvalue fundamental equation of the second-order perturbation theory which is given as (Habtamu, 2019).

$$E_n{}^2 = \sum_{n \neq m} \frac{|\langle \psi_m{}^0 \,|\, H' \,|\, \psi_n{}^0 \rangle|^2}{E_n{}^0 - E_m{}^0} \tag{13}$$

$\psi_m{}^0$ and $\psi_n{}^0$in Equation (13) represents the eigenstates for $m^{\text{th}} and \; n^{\text{th}} \; ingers$, where $E_n{}^0 and \; E_m{}^0$ are the ground state energy of a harmonic oscillator for $m^{\text{th}} and \; n^{\text{th}} \; ingers$ and H' is the perturbation Hamiltonian.

Solving the numerator of the function in Equation (13) separately which is:

$$\langle \psi_m{}^0 \,|\, H' \,|\, \psi_n{}^0 \rangle \tag{14}$$

Hence substituting Equation (2) into Equation (14) we have that;

$$\langle \psi_m{}^0 \,|\, H' \,|\, \psi_n{}^0 \rangle = \lambda_1 \tag{15}$$

where $\psi_m{}^0 = m$ and $\psi_n{}^0 = n$; in which $m \; and \; n$ are the m^{th} and n^{th} eigenfunction. Therefore, substituting Equation (8) into Equation (15), hence we have;

$$\langle \psi_m{}^0 \,|\, H' \,|\, \psi_n{}^0 \rangle = \lambda_1 \sqrt{\frac{\hbar}{2m\omega}} \langle m \,|\, a^\dagger + \; a \,|\, n \rangle \tag{16}$$

Applying the operator properties in Equation (10a) and (10b) into Equation (16), thus having that;

$$\langle \psi_m{}^0 \,|\, H' \,|\, \psi_n{}^0 \rangle = \lambda_1 \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n+1} \, \langle m \,|\, n+1 \rangle + \sqrt{n} \, \langle m \,|\, n-1 \rangle \right) \tag{17}$$

Applying the Kronecker delta function, it is important to note from Equation (17) that;

$$\langle m \,|\, n+1 \rangle = \delta_{m,n+1} \tag{18a}$$

$$\langle m \,|\, n-1 \rangle = \delta_{m,n-1} \tag{18b}$$

Therefore, substituting Equation (18a) and (18b) into Equation (17), thus we have;

$$\langle \psi_m{}^0 \,|\, H' \,|\, \psi_n{}^0 \rangle = \lambda_1 \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n+1} \, \delta_{m,n+1} + \sqrt{n} \, \delta_{m,n-1} \right) \tag{19}$$

Hence putting Equation (19) into Equation (13), Equation (20) becomes;

$$E_n{}^2 = \sum_{n \neq m} \frac{\left| \lambda_1 \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n+1} \, \delta_{m,n+1} + \sqrt{n} \, \delta_{m,n-1} \right) \right|^2}{E_n{}^0 - E_m{}^0} \tag{20}$$

here $E_n{}^0 and \;\; E_m{}^0$ are the ground state energy of a harmonic oscillator given by;

$$E_n{}^0 = \left(n + \tfrac{1}{2} \right) \hbar\omega \tag{21a}$$

$$E_m{}^0 = \left(m + \tfrac{1}{2} \right) \hbar\omega \tag{21b}$$

$$E_n{}^2 = \frac{\lambda_1{}^2}{2m\omega^2} \sum_{n \neq m} \frac{\left| \sqrt{n+1} \, \delta_{m,n+1} + \sqrt{n} \, \delta_{m,n-1} \right|^2}{n - m} \tag{22}$$

If $m = n+1; \; m = n-1$ we obtain that;

$$E_n{}^2 = \frac{\lambda_1{}^2}{2m\omega^2} \tag{23}$$

Therefore Equation (23) is the second-order energy correction to the eigenvalue E_n of a one-dimensional

quantum harmonic oscillator with a linear perturbation potential. Thus, to get the complete eigenvalue of the system. The general expression of energy function due to H' for any perturbation is given by;

$$E_n = {E_n}^0 + {E_n}^1 + {E_n}^2 \tag{24}$$

Therefore, inserting Equation (21a), (12) and (23) into Equation (24) to obtain the general expression of the energy correction. That is;

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega + \frac{{\lambda_1}^2}{2m\omega^2} \tag{25}$$

where n is a positive integer ($n = 0, 1, 2, 3 \ldots\ldots$). Therefore Equation (25) gives the exact correction to the energy and also the complete eigenvalues for the first and second approximation of one-dimensional quantum harmonic oscillator perturbed by a linear potential.

III. ONE DIMENSIONAL QUANTUM HARMONIC OSCILLATOR WITH QUADRATIC PERTURBATION POTENTIAL

Now consider a quantum harmonic oscillator with unperturbed Hamiltonian (H_0), which is already stated in Equation (1). Suppose that the system is perturbed with a quadratic potential so that;

$$H' = \lambda_2 x^2 \tag{26}$$

Equation (26) is added to the unperturbed Hamiltonian (H_0), in Equation (3). Hence we have that;

$$H = H_0 + H' = \frac{P^2}{2m} + \lambda_2 x^2 \tag{27}$$

At this point, we seek to determine the correction of the first and second-order energy by applying the first and second-order energy shift formulas.

A. First Order Energy Shift

Since the formula for the first-order energy correction in Equation (5) is; ${E_n}^1 = \langle {\psi_n}^0 \,|\, H' \,|\, {\psi_n}^0 \rangle$. Thus, putting Equation (26) into Equation (5), we have that;

$${E_n}^1 = \lambda_2 \langle {\psi_n}^0 \,|\, x^2 \,|\, {\psi_n}^0 \rangle \tag{28}$$

If ${\psi_n}^0 = n$ in which n is the n^{th} eigenfunction then;

$${E_n}^1 = \lambda_2 \langle n \,|\, x^2 \,|\, n \rangle \tag{29}$$

Applying the ladder operators stated in Equation (8) that is;

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left[a^\dagger + a\right] \text{ but for } x^2 \text{ we have that}$$

$$x^2 = \frac{\hbar}{2m\omega}\left[a^\dagger + a\right]^2 \tag{30}$$

Hence inserting Equation (30) into Equation (29), we now have that;

$${E_n}^1 = \lambda_2 \left(\frac{\hbar}{2m\omega}\right) \langle n \,|\, \left[a^\dagger + a\right]^2 \,|\, n \rangle \tag{31}$$

Expanding the ladder operator in Equation (31) we then have that;

$${E_n}^1 = \frac{\lambda_2 \hbar}{2m\omega} \langle n \,|\, a^\dagger a^\dagger + aa + a^\dagger a + aa^\dagger \,|\, n \rangle \tag{32}$$

Applying the operator properties in Equation (10a) and (10b) we now have that;

$$\begin{aligned} {E_n}^1 \quad = \quad & \frac{\lambda_2 \hbar}{2m\omega}\left(\sqrt{(n+1)(n+2)}\,\langle n \,|\, n+2\rangle \right. \\ & + \sqrt{(n-1)n}\,\langle n \,|\, n-2\rangle + 2n \\ & \left. + \; 1\langle n \,|\, n\rangle\right) \end{aligned} \tag{33}$$

Applying the orthogonality condition, we have;

$${E_n}^1 = \frac{\lambda_2 \hbar}{m\omega}\left(n + \frac{1}{2}\right) \tag{34}$$

Therefore Equation (34) is the first-order energy correction to the eigenvalue E_n of a one-dimensional quantum harmonic oscillator.

B. Second-Order Energy Shift

Here we seek to obtain the second-order energy correction using the eigenvalue fundamental equation of the second-order perturbation theory which is already stated in Equation

(13); substituting Equation (26) into the numerator of Equation (13) we the have;

$$\langle \psi_m{}^0 \,|\, H' \,|\, \psi_n{}^0 = \lambda_2 \langle m \,|\, x^2 \,|\, n \rangle \ \rangle \tag{35}$$

Therefore, substituting Equation (30) into Equation (35) hence we have that;

$$\langle \psi_m{}^0 \,|\, H' \,|\, \psi_n{}^0 \rangle = \lambda_2 \left(\frac{\hbar}{2m\omega} \right) \langle m \,|\, \left[a^\dagger + a \right]^2 \,|\, n \rangle \tag{36}$$

Expanding the ladder operator in Equation (36) hence we have that;

$$= \frac{\lambda_2 \hbar}{2m\omega} \langle m \,|\, a^\dagger a^\dagger + aa + a^\dagger a + aa^\dagger \,|\, n \rangle \tag{37}$$

Applying the operator properties in Equation (10a) and (10b) into Equation (32), thus having that;

$$\langle m \,|\, H' \,|\, n \rangle = \frac{\lambda_2 \hbar}{2m\omega} \left(\sqrt{(n+1)(n+2)}\, \delta_{m,n+2} + \sqrt{(n-1)n}\, \delta_{m,n-2} + (2n+1)\delta_{m,n} \right) \tag{38}$$

Inserting Equation (38) into Equation (13) then we have that;

$$E_n{}^2 = \sum_{n \neq m} \frac{\left| \frac{\lambda_2 \hbar}{2m\omega} \left(\sqrt{(n+1)(n+2)}\, \delta_{m,n+2} + \sqrt{(n-1)n}\, \delta_{m,n-2} + (2n+1)\delta_{m,n} \right) \right|^2}{E_n{}^0 - E_m{}^0} \tag{39}$$

From Equation (39), the first term contributes when $m = n + 2$, the second term contributes when $m = n - 2$, but the third term contributes only when $m = n$, which is excluded in our relation. Hence substituting Equation (21a) into Equation (39), we get;

$$E_n{}^2 = \frac{\lambda_2{}^2 \hbar}{4m^2\omega^3} \left[-\frac{1}{2}(n+1)(n+2) + \frac{1}{2} n(n-1) \right] \tag{40}$$

Solving Equation (40) further we finally obtain that;

$$E_n{}^2 = -\frac{\lambda_2{}^2 \hbar}{2m^2\omega^3} \left(n + \frac{1}{2} \right) \tag{41}$$

Therefore Equation (41) is the second-order energy correction to the eigenvalue E_n of a one-dimensional quantum harmonic oscillator with a quadratic perturbation potential. Hence, inserting Equation (21a), (34) and (41) into Equation (24) to get the complete eigenvalue of the system. The general expression of energy function due to H' for any perturbation now becomes;

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right) + \frac{\lambda_2 \hbar}{m\omega} \left(n + \frac{1}{2} \right) - \frac{\lambda_2{}^2 \hbar}{2m^2\omega^3} \left(n + \frac{1}{2} \right) \tag{42}$$

Equation (42) can be solved further to obtain that;

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right) \left[1 + \frac{\lambda_2}{m\omega^2} + \frac{\lambda_2{}^2}{2m^2\omega^4} \right] \tag{43}$$

where n is a positive integer *i. e.* ($n = 0, 1, 2, 3 \ldots\ldots$). Therefore Equation (43) gives the exact correction to the energy and also the complete eigenvalues for the first and second approximation of one-dimensional quantum harmonic oscillator perturbed by a quadratic potential.

IV. ONE DIMENSIONAL QUANTUM HARMONIC OSCILLATOR WITH CUBIC PERTURBATION POTENTIAL

This time we will consider a cubic potential perturbing on the same unperturbed Hamiltonian(H_0). The cubic perturbation potential is given as;

$$H' = \lambda_3 x^3 \tag{44}$$

Inserting Equation (44) and Equation (1) into Equation (3) so that the total Hamiltonian become;

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 x^2 + \lambda_3 x^3 \tag{45}$$

To solve for the correction of the first-order energy and second-order energy by using the first order and second order energy shift formula, we adopted the method of perturbation theory.

A. First Order Energy Shift

Adopting Equation (5) which is $E_n{}^1 = \langle \psi_n{}^0 \,|\, H' \,|\, \psi_n{}^0 \rangle$. Thus, putting Equation (44) into the equation we have that;

$$E_n{}^1 = \lambda_3 \langle \psi_n{}^0 \,|\, x^3 \,|\, \psi_n{}^0 \rangle \tag{46}$$

If $\psi_n{}^0 = n$ in which n is the n^{th} eigenfunction, therefore;

$$E_n{}^1 = \lambda_3 \langle n \,|\, x^3 \,|\, n \rangle \tag{47}$$

Applying the ladder operators stated in Equation (8) that is;

$x = \sqrt{\frac{\hbar}{2m\omega}}\,[a^\dagger + a]$, but for x^3 we have that

$$x^3 = \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} [a^\dagger + a]^3 \tag{48}$$

Hence inserting Equation (48) into Equation (47), we now have that;

$$E_n{}^1 = \lambda_3 \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \langle n \,|\, [a^\dagger + a]^3 \,|\, n \rangle \tag{49}$$

Expanding the ladder operator in Equation (49) we then have that;

$$\begin{aligned} E_n{}^1 = \lambda_3 \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \langle n \,|\, & a^\dagger a^\dagger a^\dagger + aaa^\dagger + a^\dagger a^\dagger a + aa^\dagger a^\dagger \\ & + a^\dagger a^\dagger a + aaa + a^\dagger aa \\ & + aaa^\dagger \,|\, n \rangle \end{aligned} \tag{50}$$

$$\begin{aligned} E_n{}^1 = \lambda_3 \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \big(& \sqrt{(n+1)(n+2)(n+3)}\,\langle n \,|\, n+3 \rangle \\ & + (n+1)\sqrt{n}\,\langle n \,|\, n-1 \rangle \\ & + (n+1)\sqrt{n+1}\,\langle n \,|\, n+1 \rangle \\ & + (n+2)\sqrt{n+1}\,\langle n \,|\, n+1 \rangle \\ & + n\sqrt{n+1}\langle n \,|\, n+1 \rangle \\ & + \sqrt{n(n-1)(n-2)}\langle n \,|\, n-3 \rangle + (n \\ & - 1)\sqrt{n}\,\langle n \,|\, n-1 \rangle \\ & + n\sqrt{n}\,\langle n \,|\, n-1 \rangle \big) \end{aligned} \tag{51}$$

Applying again the orthogonality condition; Equation (51) reduces to;

$$E_n{}^1 = 0 \tag{52}$$

Therefore Equation (52) is the first-order energy correction to the eigenvalue E_n of a one-dimensional quantum harmonic oscillator with a cubic perturbation potential.

B. Second-Order Energy Shift

At this point, we seek to obtain the second-order energy correction using the eigenvalue fundamental equation of the second-order perturbation theory as stated in Equation (13).

$$\langle \psi_m{}^0 \,|\, H' \,|\, \psi_n{}^0 \rangle = \lambda_3 \,\langle m \,|\, x^3 \,|\, n \rangle \tag{53}$$

Therefore, substituting Equation (48) into Equation (53) hence we have that;

$$\langle m \,|\, H' \,|\, n \rangle = \lambda_3 \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \langle m \,|\, [a^\dagger + a]^3 \,|\, n \rangle \tag{54}$$

From the expression obtained in Equation (53) if we substitute the ket function Equation (51) can be rewritten as;

$$\begin{aligned} \langle m \,|\, H' \,|\, n \rangle = \lambda_3 \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \big(& \sqrt{(n+1)(n+2)(n+3)}\,\delta_{m,n+3} \\ & + (n+1)\sqrt{n}\,\delta_{m,n-1} \\ & + (n+1)\sqrt{n+1}\,\delta_{m,n+1} \\ & + (n+2)\sqrt{n+1}\,\delta_{m,n+1} + n\sqrt{n+1}\delta_{m,n+1} \\ & + \sqrt{(n-2)(n-1)n}\delta_{m,n-3} + (n \\ & - 1)\sqrt{n}\;\delta_{m,n-1} + n\sqrt{n}\,\delta_{m,n-1} \big) \end{aligned} \tag{55}$$

Contributions of these terms in Equation (53) lies when $m = n+3, m = n-1, m = n = 1\,, m = n-3$; hence it will be more appreciable to solve these relations separately to obtain an appropriate function thus;

Case I: $m = n + 3$

$$\begin{aligned} \left|\langle n+3 \,|\, H' \,|\, n \rangle\right|^2 &= \left|\lambda_3 \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \sqrt{(n+1)(n+2)(n+3)}\right|^2 \\ &= {\lambda_3}^2 \left(\frac{\hbar}{2m\omega}\right)^3 (n+1)(n+2)(n \\ &\quad + 3) \end{aligned} \tag{56}$$

Case II: $m = n - 1$. Since it has three terms in the relation, hence we will obtain the single relation by adding the terms up so therefore we have that;

$$\left|\langle n-1\left|H'\right|n\rangle\right|^2 = \left|\lambda_3\left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} n + 1\sqrt{n} + n - 1\sqrt{n} + n\sqrt{n}\right|^2 = {\lambda_3}^2\left(\frac{\hbar}{2m\omega}\right)^3 9n^3 \quad (57)$$

Case III: $m = n + 1$. Since it has three terms in the relation, hence we will obtain the single relation by adding the terms up so therefore we have that;

$$\left|\langle n+1\left|H'\right|n\rangle\right|^2 = \left|\lambda_3\left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} n + 1\sqrt{n+1} + n + 2\sqrt{n+1} + n\sqrt{n} + 1\right|^2 = {\lambda_3}^2\left(\frac{\hbar}{2m\omega}\right)^3 9(n+1)^3 \quad (58)$$

Case IV: $m = n - 3$

$$\left|\langle n-3\left|H'\right|n\rangle\right|^2 = \left|\lambda_3\left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\sqrt{(n-2)(n-1)n}\right|^2 = {\lambda_3}^2\left(\frac{\hbar}{2m\omega}\right)^3 (n-2)(n-1)n \quad (59)$$

Hence combining the contributions from all these terms obtained from Equation (56) to Equation (59) substituting them into Equation (13) we then have;

$${E_n}^2 = \frac{{\lambda_3}^2\left(\frac{\hbar}{2m\omega}\right)^3[(n+1)(n+2)(n+3)+(n-2)(n-1)n+9(n+1)^3+9n^3]}{{E_n}^0-{E_m}^0} \quad (60)$$

Substituting Equation (21a) and (21b) into Equation (60) we obtain;

$${E_n}^2 = {\lambda_3}^2\left(\frac{\hbar}{2m\omega}\right)^3\frac{1}{\hbar\omega}\left[-\frac{1}{3}(n+1)(n+2)(n+3) + \frac{1}{3}(n-2)(n-1)n - 9(n+1)^3 + 9n^3\right] \quad (61)$$

With little effort in algebra we can solve further to obtain;

$${E_n}^2 = -\frac{1}{8}\frac{{\lambda_3}^2\hbar^2}{m^3\omega^4}\left[30\left(n+\frac{1}{2}\right)^2 + \frac{7}{2}\right] \quad (62)$$

Therefore Equation (62) is the correction to the eigenvalue E_n of a one-dimensional quantum harmonic oscillator with a cubic perturbation potential. Thus, inserting Equation (21a), (52) and (62) into Equation (24) to get the complete eigenvalue of the system. The general expression of energy function due to H' for any perturbation now becomes;

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) - \frac{1}{8}\frac{\lambda_3\hbar^2}{m^3\omega^4}\left[30\left(n+\frac{1}{2}\right)^2 + \frac{7}{2}\right] \quad (63)$$

where n is a positive integer *i. e.* ($n = 0, 1, 2, 3 \ldots\ldots$). Therefore Equation (63) gives the exact correction to the energy and also the complete eigenvalues for the first and second approximation of one-dimensional quantum harmonic oscillator perturbed by a cubic potential.

V. ONE DIMENSIONAL QUANTUM ANHARMONIC OSCILLATOR WITH POLYNOMIAL PERTURBATION POTENTIAL

This time we will consider a polynomial potential perturbing on the same unperturbed Hamiltonian (H_0) as stated in Equation (1). Polynomial perturbation potential for integral values of k; where k, the order of the polynomial, is a positive integer *i. e.* ($k = 0, 1, 2, 3 \ldots\ldots$). as given in (Barry S, 1991);

$$H' = \lambda_k x^k \quad (64)$$

Equation (64) is the generalised perturbation. Hence, inserting it and Equation (1) into Equation (3) so that the total Hamiltonian will now become;

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 + \lambda_k x^k \quad (65)$$

Similarly, to solve for the correction of the first-order energy and second-order energy by using the first order and second order energy shift formula, we adopted the method of perturbation theory.

A. First Order Energy Shift

Adopting Equation (5) which is ${E_n}^1 = \langle {\psi_n}^0\left|H'\right|{\psi_n}^0\rangle$. Thus, putting Equation (63) into the equation we have that;

$${E_n}^1 = \lambda_k\langle {\psi_n}^0\left|x^k\right|{\psi_n}^0\rangle \quad (66)$$

If ${\psi_n}^0 = n$ in which n is the n^{th} eigenfunction, therefore;

$$E_n{}^1 = \lambda_k \langle n \left| x^k \right| n \rangle \quad (67)$$

Applying the ladder operators stated in Equation (8) that is;

$x = \sqrt{\frac{\hbar}{2m\omega}}\left[a^\dagger + a\right]$ but for x^k we have that

$$x^k = \left(\frac{\hbar}{2m\omega}\right)^{\frac{k}{2}}\left[a^\dagger + a\right]^k \quad (68)$$

Inserting Equation (68) into Equation (67), we now have that;

$$E_n{}^1 = \lambda_k \left(\frac{\hbar}{2m\omega}\right)^{\frac{k}{2}} \langle n \left| \left[a^\dagger + a\right]^k \right| n \rangle \quad (69)$$

Solving the ladder operator in Equation (69) which is a binomial expansion of the power k we then have that;

$$\left[a^\dagger + a\right]^k = a^k + {}^kC_1 a^\dagger a^{k-1} + .. + {}^kC_r a^{\dagger r} a^{k-r} + \cdots + a^{\dagger k} \quad (70)$$

For any integral values of k as given in (Pathak, 2000). Is expressed as;

$$\left[a^\dagger + a\right]^k = \sum_{r=0,2,4\ldots}^{k} {t_r}^k \, C_r \left(a^\dagger + a\right)^{k-r} \quad (71)$$

$$t_r = \frac{(r-1)!}{2^{\left(\frac{r}{2}-1\right)}\left(\frac{r}{2}-1\right)!} \quad for\ r \geq 4 \quad and\ t_0 = t_2 = 1 \quad (72)$$

$$E_n{}^1 = \lambda_k \left(\frac{\hbar}{2m\omega}\right)^{\frac{k}{2}} \langle n \left| \sum_{r=0,2,4\ldots}^{k} {t_r}^k \, C_r \left(a^\dagger + a\right)^{k-r} \right| n \rangle \quad (73)$$

$$= \lambda_k \left(\frac{\hbar}{2m\omega}\right)^{\frac{k}{2}} \langle n \left| \sum_{r=0,2,4\ldots}^{k} {t_r}^k \, {C_r}^{k-r} C_{\frac{k-r}{2}} \, a^{\dagger \frac{k-r}{2}} a^{\frac{k-r}{2}} \right| n \rangle$$

$$= \lambda_k \left(\frac{\hbar}{2m\omega}\right)^{\frac{k}{2}} \sum_{r=0,2,4\ldots}^{k} {t_r}^k \, {C_r}^{k-r} {C_{\frac{k-r}{2}}}^n C_{\frac{k-r}{2}} \left(\frac{k-r}{2}\right)! \quad (74)$$

Equation (74) is the total correction to the energy eigenvalues for the first approximation of a one-dimensional quantum anharmonic oscillator perturbed by a polynomial potential. It involves summations that are more easily to evaluate.

B. Second-Order Energy Shift

At this point, we seek to obtain the second-order energy correction using the eigenvalue fundamental equation of the second-order perturbation theory as stated in Equation (13). Inserting Equation (64) into the numerator function of Equation (13). Thus, we then have that;

$$\langle {\psi_m}^0 \left| H' \right| {\psi_n}^0 \rangle = \lambda_k \langle m \left| x^k \right| n \rangle \quad (75)$$

Therefore, substituting Equation (68) into Equation (75) we have that;

$$\langle m \left| H' \right| n \rangle = \lambda_k \left(\frac{\hbar}{2m\omega}\right)^{\frac{k}{2}} \langle m \left| \left[a^\dagger + a\right]^k \right| n \rangle \quad (76)$$

Equation (71) can be substituted into Equation (76) to obtain that;

$$\langle m \left| H' \right| n \rangle = \lambda_k \left(\frac{\hbar}{2m\omega}\right)^{\frac{k}{2}} \langle m \left| \sum_{r=0,2,4\ldots}^{k} {t_r}^k \, C_r \left(a^\dagger + a\right)^{k-r} \right| n \rangle \quad (77)$$

As functions of t_r as stated by (Pathak, 200). Thus, we can have the total correction to the energy eigenvalues for the second approximation of one-dimensional quantum anharmonic oscillator perturbed by a polynomial potential by substituting Equation (77) into (13) as;

$$E_n{}^2 = {\lambda^2}_k \left(\frac{\hbar}{2m\omega}\right)^k \sum_{n \neq m} \frac{\left| \langle m \right| \sum_{r=0,2,4\ldots}^{k} {t_r}^k C_r \left(a^\dagger + a\right)^{k-r} \left| n \right\rangle \right|^2}{{E_n}^0 - {E_m}^0} \quad (78)$$

Equation (78) can be rewritten by substituting Equation (21a) & (21b).

$$E_n{}^2 = {\lambda^2}_k \left(\frac{\hbar}{2m\omega}\right)^k \sum_{n \neq m} \frac{\left| \langle m \right| \sum_{r=0,2,4\ldots}^{k} {t_r}^k C_r \left(a^\dagger + a\right)^{k-r} \left| n \right\rangle \right|^2}{\left(n+\frac{1}{2}\right)\hbar\omega \ - \left(m+\frac{1}{2}\right)\hbar\omega} \quad (79)$$

Solving further we have;

$$E_n{}^2 = {\lambda^2}_k \left(\frac{\hbar}{2m\omega}\right)^k \sum_{n \neq m} \frac{\left| \langle m \right| \sum_{r=0,2,4\ldots}^{k} {t_r}^k C_r \left(a^\dagger + a\right)^{k-r} \left| n \right\rangle \right|^2}{\hbar\omega(n-m)} \quad (80)$$

Therefore, Equation (80) is the second-order energy correction to the eigenvalue E_n of a one-dimensional quantum anharmonic oscillator with a polynomial perturbation potential. Thus, inserting Equation (21a), (75)

and (80) into Equation (24), we get the complete eigenvalue of the system. The general expression of energy function due to H' for any perturbation now becomes;

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) + \lambda_k\left(\frac{\hbar}{2m\omega}\right)^{\frac{k}{2}}\sum_{r=0,2,4\ldots}^{k} {t_r}^k\, {C_r}^{k-r}\, C_{\frac{k-r}{2}}{}^{n}\, C_{\frac{k-r}{2}}\left(\frac{k-r}{2}\right)! + \lambda^2{}_k\left(\frac{\hbar}{2m\omega}\right)^{k}\sum_{n\neq m}\frac{|\langle m|\sum_{r=0,2,4\ldots}^{k} t_r{}^k C_r\left(a^{\dagger}+a\right)^{k-r}|n\rangle|^2}{\hbar\omega(n-m)} \quad (81)$$

where n, m and k are positive integers $i.e.(n,m,k = 0,1,2,3\ldots..)$. Therefore Equation (82) gives the exact correction to the energy and also the complete eigenvalues for the first and second approximation of one-dimensional quantum anharmonic oscillator perturbed by a polynomial potential.

VI. CONCLUSION

In this work, we have reported the solutions of one-dimensional quantum harmonic and anharmonic oscillator perturbed by a linear, quadratic, cubic and polynomial potential using the time-independent perturbation theory method. We presented the correction to first and second-order energy using the first and second-order energy shift equations. From the analysis, we observed that the first-order energy eigenvalues for linear and cubic potential are zero, but non-zero for a quadratic potential. The general expression of the energy eigenvalues due to H' for the linear, quadratic, cubic and polynomial potential was obtained, and the values of the energy function depend on the values of n and the order of polynomial k. The solutions could find applications in nonlinear deterministic equations encountered in quantum field theory and the qubit when quantum degrees of freedom in a potential well is either bound, free or scattered. More so, the solutions of the quantum anharmonic oscillator could be used to describe as well as predict the time evolution of quasiclassical systems employed in modelling equations describing stochastic processes such as probability distributions of stock price return which dynamics of its movement is considered an analogue of the motion of a quantum particle.

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VIII. REFERENCES

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