The Symmetries of \((k: \alpha_1, \alpha_2, \ldots, \alpha_k)\)-step Fibonacci Functions

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It is well known that the Fibonacci sequence \((F_n)\) is denoted by \(F_0 = 0, F_1 = 1\) and \(F_n = F_{n-1} + F_{n-2}\), while the Lucas sequence \((L_n)\) is denoted by \(L_0 = 2, L_1 = 1\) and \(L_n = L_{n-1} + L_{n-2}\). There are several studies showing relations between these two sequences. An interesting generalisation of both the sequences is a Fibonacci function \(f: \mathbb{R} \rightarrow \mathbb{R}\) defined by \(f(x + 2) = f(x + 1) + f(x)\) for any real number \(x\) (Elmore, 1967). Research about periods of Fibonacci numbers modulo \(m\) (Jameson, 2018) results in a contribution on the existence of primitive period of a Fibonacci function \(f: \mathbb{Z} \rightarrow \mathbb{Z}\) modulo \(m\) (Thongngam & Chinram, 2019). Recently, a \(k\)-step Fibonacci function \(f: \mathbb{Z} \rightarrow \mathbb{Z}\) denoted by \(f(n + k) = f(n + k - 1) + f(n + k - 2) + \cdots + f(n)\) for any integer \(n\) and \(k \geq 2\) (which is a generalisation of a Fibonacci function \(f: \mathbb{Z} \rightarrow \mathbb{Z}\)) is introduced and the existence of primitive period of this function modulo \(m\) is established (Tongron & Kerdmongkon, 2022). In this work, let \(k\) be an integer \(\geq 2\). For nonnegative integers \(\alpha_1, \alpha_2, \ldots, \alpha_k\) and \(\alpha_1 \neq 0\), a \((k: \alpha_1, \alpha_2, \ldots, \alpha_k)\)-step Fibonacci function \(f: \mathbb{Z} \rightarrow \mathbb{Z}\) is defined by \(f(n) = f(n - \alpha_1) + f(n - \alpha_1 - \alpha_2) + \cdots + f(n - \alpha_1 - \alpha_2 - \cdots - \alpha_k)\) for any integer \(n\). In fact, a \(k\)-step Fibonacci function is a special case of a \((k: \alpha_1, \alpha_2, \ldots, \alpha_k)\)-step Fibonacci function. We present the existence of primitive period of this function modulo \(m\) and show that certain \((k: \alpha_1, \alpha_2, \ldots, \alpha_k)\)-step Fibonacci functions are symmetric-like.

**Keywords:** Fibonacci functions; \(k\)-step Fibonacci function; \((k: \alpha_1, \alpha_2, \ldots, \alpha_k)\)-step Fibonacci function; primitive period modulo \(m\); symmetric-like

1. **INTRODUCTION**

The Fibonacci sequence \((F_n)\) is defined by (Koshy, 2001; Vorob’ev, 2011):

\[ F_0 = 0, F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \]

for any natural number \(n \geq 2\). The beginning of the sequence is thus:

\[ 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots \]

Similar to the Fibonacci sequence, the Lucas sequence \((L_n)\) is defined by (Koshy, 2001):

\[ L_0 = 2, L_1 = 1 \text{ and } L_n = L_{n-1} + L_{n-2} \]

for any natural number \(n \geq 2\). The beginning of the sequence is thus:

\[ 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, \ldots \]

Recently, there are several interesting relations between the Fibonacci sequence and the Lucas sequence, for example, (Adegoke, 2022; Phunphayap, Khemaratchatakumthorn & Sumrithnorrapong, 2022), etc.

In 1967, Elmore (Elmore, 1967) consider a relation between the Fibonacci sequence and the Lucas sequence and define a Fibonacci function \(f: \mathbb{R} \rightarrow \mathbb{R}\) which is denoted by:

\[ f(x + 2) = f(x + 1) + f(x) \]

for all real numbers \(x\). Observe that if \(f(0) = 0\) and \(f(1) = 1\), then we get the Fibonacci sequence. Furthermore, if \(f(0) = 2\) and \(f(1) = 1\), then we get the Lucas sequence. Consequently, a Fibonacci function is a generalisation of both the Fibonacci sequence and the Lucas sequence.

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In 2018, Jameson (Jameson, 2018) studies periods of Fibonacci numbers modulo $m$ and provides some properties on periods of such numbers. His work motivates Thongngam and Chinram (Thongngam & Chinram, 2019) to show the existence of primitive period of a Fibonacci function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ modulo $m$. They also establish some relations among periods and the primitive periods of such functions.

Recently, Tongron and Kerdmongkon (Tongron & Kerdmongkon, 2022) study about a $k$-step Fibonacci function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by:

$$f(n + k) = f(n + k - 1) + f(n + k - 2) + \cdots + f(n)$$

for any integer $n$ and $k \geq 2$. We can say equivalently that it is denoted by:

$$f(n) = f(n - 1) + f(n - 2) + \cdots + f(n - k)$$

for any integer $n$ and $k \geq 2$. Observe that this function when $k = 2$ is a generalisation of a Fibonacci function defined from $\mathbb{Z}$ to $\mathbb{Z}$. We refer to their work as follows:

**Theorem 1.1.** (Tongron & Kerdmongkon, 2022) Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be a $k$-step Fibonacci function and $m$ be a positive integer $> 1$. Then there exists an integer $1 \leq l \leq m^k$ such that $f(n + l) \equiv f(n) \pmod{m}$ for any integer $n$.

Such integer $l$ is called a Period of $f$ modulo $m$. If such integer $l$ is the smallest, then it is called the Primitive Period of $f$ modulo $m$ and write $l = \ell_f(m)$.

**Theorem 1.2.** (Tongron & Kerdmongkon, 2022) Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be a $k$-step Fibonacci function and $l$, $m$ be positive integers $> 1$. $l$ is a period of $f$ modulo $m$ if and only if $\ell_f(m) \mid l$.

**Theorem 1.3.** (Tongron & Kerdmongkon, 2022) Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be a $k$-step Fibonacci function and $m$, $n$ be positive integers $> 1$. If $\gcd(m, n) = 1$, then $\ell_f(mn) = \text{lcm}[\ell_f(m), \ell_f(n)]$.

Indeed, Thongngam and Chinram’s results (Thongngam & Chinram, 2019) are special cases of the above facts. Tongron and Kerdmongkon (Tongron & Kerdmongkon, 2022) also provide the explicit primitive periods of some $k$-step Fibonacci function as follows:

**Lemma 1.4.** (Tongron & Kerdmongkon, 2022) Let $m$ be a positive integer and $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be a $k$-step Fibonacci function with the starting values $f(0) = a_0, f(1) = a_1, \ldots, f(k - 1) = a_{k-1}$ and $\gcd(m, k - 1) = 1$. Then $m|a_i$ for all $i \in \{0, 1, \ldots, k - 1\}$ if and only if $\ell_f(m) = 1$.

**Theorem 1.5.** (Tongron & Kerdmongkon, 2022) Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be a 2-step Fibonacci function with the starting values $f(0) = a$ and $f(1) = b$. Assume that $2m \nmid a$ or $2m \nmid b$. For a positive integer $m$, $m|a$ and $m|b$ if and only if $\ell_f(2m) = 3$.

**Theorem 1.6.** (Tongron & Kerdmongkon, 2022) Let $m$ be a positive odd integer and $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be a 3-step Fibonacci function with the starting values $f(0) = a$, $f(1) = b$ and $f(2) = c$. Assume that $3m \nmid a$, $3m \nmid b$ or $3m \nmid c$. Then the following statements hold.

1. If $m|a$, $m|b$ and $m|c$ then $\ell_f(3m) = 13$.
2. If $\ell_f(3m) = 13$, then
   - $91a + 141b + 168c \equiv 0 \pmod{m}$
   - $168a + 259b + 309c \equiv 0 \pmod{m}$
   - $309a + 477b + 568c \equiv 0 \pmod{m}$.

**Corollary 1.7.** (Tongron & Kerdmongkon, 2022) Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be a 3-step Fibonacci function with the starting values $f(0) = a$, $f(1) = b$ and $f(2) = c$. Then the following statements hold.

1. If $\ell_f(9) = 13$ and $a, b$ or $c$ is not divisible by $9$, then $3|a$, $3|b$ and $3|c$.
2. If $\ell_f(21) = 13$ and $a, b$ or $c$ is not divisible by $21$, then $7|a$, $7|b$ and $7|c$.

**Theorem 1.8.** (Tongron & Kerdmongkon, 2022) Let $m$ be a positive integer and $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be a 4-step Fibonacci function with the starting values $f(0) = a$, $f(1) = b$, $f(2) = c$ and $f(3) = d$. Assume that $\gcd(4m, 3) = 1$ and $a, b, c$ or $d$ is not divisible by $4m$. Then the following statements hold.

1. If $m|a$, $m|b$, $m|c$ and $m|d$, then
   $$\ell_f(4m) = \begin{cases} 5 & \text{if } b + c + d, a + b + 2c + 2d, 2a + 3b + 3c + 4d \\ 10 & \text{and } 4a + 6b + 7c + 7d \text{ are divisible by } 2m, \\ \text{otherwise}. \end{cases}$$
2. If $\ell_f(4m) = 10$, then
In this paper, we define a \((k: \alpha_1, \alpha_2, ..., \alpha_k)\)-step Fibonacci function \(f: \mathbb{Z} \rightarrow \mathbb{Z}\) for nonnegative integers \(\alpha_1, \alpha_2, ..., \alpha_k\) and \(\alpha_1 \neq 0\) by:

\[
f(n) = f(n - \alpha_1) + f(n - \alpha_1 - \alpha_2) + \cdots + f(n - \alpha_1 - \alpha_2 - \cdots - \alpha_k)
\]

for any integer \(n\). Notice that the \((k: \alpha_1, \alpha_2, ..., \alpha_k)\)-step Fibonacci function is a generalisation of a \(k\)-step Fibonacci function when all \(\alpha\) are equal to 1. Theorem 1.1 – 1.3 are going to be proven in the version of \((k: \alpha_1, \alpha_2, ..., \alpha_k)\)-step Fibonacci functions. There are also several examples to support our facts. Some of these examples motivate us to verify that certain \((k: \alpha_1, \alpha_2, ..., \alpha_k)\)-step Fibonacci functions are symmetric-like.

II. MAIN RESULTS

Let \(k\) be an integer \(\geq 2\) and \(\alpha_1, \alpha_2, ..., \alpha_k\) be nonnegative integers such that \(\alpha_1 \neq 0\). Recall that a \((k: \alpha_1, \alpha_2, ..., \alpha_k)\)-step Fibonacci function \(f: \mathbb{Z} \rightarrow \mathbb{Z}\) is defined by:

\[
f(n) = f(n - \alpha_1) + f(n - \alpha_1 - \alpha_2) + \cdots + f(n - \alpha_1 - \alpha_2 - \cdots - \alpha_k)
\]

for any integer \(n\). For general use, a \((k: \alpha_1, \alpha_2, ..., \alpha_k)\)-step Fibonacci function \(f: \mathbb{Z} \rightarrow \mathbb{Z}\) satisfies:

\[
f(n) = f(n + \alpha_1 + \alpha_2 + \cdots + \alpha_k) - f(n + \alpha_2 + \cdots + \alpha_k) - f(n + \alpha_3 + \cdots + \alpha_k) - \cdots - f(n + \alpha_k)
\]

for any integer \(n\).

**Example 2.1.** Let \(f: \mathbb{Z} \rightarrow \mathbb{Z}\) be a \((3: 2,1,1)\)-step Fibonacci function such that \(f(0) = 0\), \(f(1) = 1\), \(f(2) = -1\) and \(f(3) = -2\). We can calculate the other \(f(n)\) as follow:

\[
\begin{align*}
\quad f(-3) & = f(1) - f(-1) - f(-2) = 2 \\
f(-2) & = f(2) - f(0) - f(-1) = 2 \\
f(-1) & = f(3) - f(1) - f(0) = -3 \\
f(0) & = 0 \\
f(1) & = 1 \\
f(2) & = -1 \\
f(3) & = -2 \\
f(4) & = f(2) + f(1) + f(0) = 0 \\
f(5) & = f(3) + f(2) + f(1) = -2 \\
f(6) & = f(4) + f(3) + f(2) = -3 \\
& \vdots
\end{align*}
\]

Then we get the following tables:

**Table 1.** The values of the \((3: 2,1,1)\)-step Fibonacci function \(f(n)\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>(-10)</th>
<th>(-9)</th>
<th>(-8)</th>
<th>(-7)</th>
<th>(-6)</th>
<th>(-5)</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(n))</td>
<td>4</td>
<td>10</td>
<td>-7</td>
<td>-4</td>
<td>-6</td>
<td>-5</td>
<td>-4</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(n))</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td>0</td>
<td>-2</td>
<td>-3</td>
<td>-4</td>
<td>-5</td>
<td>-9</td>
</tr>
</tbody>
</table>

**Example 2.2.** Let \(f: \mathbb{Z} \rightarrow \mathbb{Z}\) be a \((4: 1,0,0,1)\)-step Fibonacci function such that \(f(0) = 0\) and \(f(1) = 1\). We can calculate the other \(f(n)\) as follow:

\[
\begin{align*}
\quad f(-3) & = f(-1) - f(-2) - f(-2) = 10 \\
f(-2) & = f(0) - f(-1) - f(-1) = -3 \\
f(-1) & = f(1) - f(0) - f(0) = 1 \\
f(0) & = 0 \\
f(1) & = 1 \\
f(2) & = f(1) + f(1) = 3 \\
f(3) & = f(2) + f(2) + f(2) = 10 \\
& \vdots
\end{align*}
\]

Then we get the following tables:

**Table 2.** The values of the \((4: 1,0,0,1)\)-step Fibonacci function \(f(n)\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>(-8)</th>
<th>(-7)</th>
<th>(-6)</th>
<th>(-5)</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(n))</td>
<td>-3927</td>
<td>1189</td>
<td>-360</td>
<td>109</td>
<td>-33</td>
<td>10</td>
<td>-3</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(n))</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>10</td>
<td>33</td>
<td>109</td>
<td>360</td>
<td>1189</td>
<td>3927</td>
<td>12970</td>
</tr>
</tbody>
</table>

Next, we show the existence of primitive period of \((k: \alpha_1, \alpha_2, ..., \alpha_k)\)-step Fibonacci functions modulo \(m\).

**Theorem 2.3.** Let \(f: \mathbb{Z} \rightarrow \mathbb{Z}\) be a \((k: \alpha_1, \alpha_2, ..., \alpha_k)\)-step Fibonacci function with \(\alpha_k \geq 1\) and \(m\) be a positive integer.
> 1. Then there exists an integer $1 \leq l \leq m^{\alpha_1 + \alpha_2 + \cdots + \alpha_k}$ such that $f(n + l) \equiv f(n) \pmod{m}$ for any integer $n$.

**Proof.** For any integer $a \in \{0, 1, \ldots, m^{\alpha_1 + \alpha_2 + \cdots + \alpha_k} + 1\}$ elements, consider \((a, a + \alpha_2 + \cdots + \alpha_k)\) - tuple 
\[
(f(a), f(a + 1), \ldots, f(a + 1 + \alpha_2 + \cdots + \alpha_k - 1))
\]
modulo $m$ which can be $m^{\alpha_1 + \alpha_2 + \cdots + \alpha_k}$ possible values:
\[
(0, 0, 0, \ldots, 0, 0), (0, 0, 0, \ldots, 0, 1), \ldots, (0, 0, \ldots, 0, 1, m - 1),
\]
\[
(0, 0, \ldots, 1, 0), (0, 0, \ldots, 1, 1), \ldots, (0, 0, \ldots, 1, m - 1),
\]
\[
\vdots \quad \vdots \quad \vdots 
\]
\[
(m - 1, m - 1, \ldots, m - 1, 0), (m - 1, m - 1, \ldots, m - 1, 1), \ldots
\]
\[
(m - 1, m - 1, \ldots, m - 1, m - 1, m - 1).
\]
We obtain from the Pigeonhole Principle (Burton, 2011) that there are integers $0 \leq i < j \leq m^{\alpha_1 + \alpha_2 + \cdots + \alpha_k}$ such that:
\[
(f(i), f(i + 1), \ldots, f(i + 1 + \alpha_2 + \cdots + \alpha_k - 1)) \equiv (f(i), f(i + 1), \ldots, f(i + 1 + \alpha_2 + \cdots + \alpha_k - 1)) \pmod{m}.
\]
In other words, we have:
\[
f(i + a) \equiv f(i + a) \pmod{m},
\]
where $a \in \{0, 1, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_k - 1\}$. Choose a positive integer $l := j - i$ so that:
\[
f(i + a + l) \equiv f(i + a) \pmod{m}
\]
and $1 \leq l \leq m^{\alpha_1 + \alpha_2 + \cdots + \alpha_k}$. The proof is divided into two cases: $n \geq i$ and $n \leq i$.

**Case 1.** Assume that $f(r + l) \equiv f(r) \pmod{m}$ for $i \leq r \leq n$ and $n \geq i + \alpha_1 + \alpha_2 + \cdots + \alpha_k - 1$. Since $i \leq n + 1 - \alpha_1 - \alpha_2 - \cdots - \alpha_k < n + 1 - \alpha_1 - \alpha_2 - \cdots - \alpha_k - 1 \leq \cdots \leq n + 1 - \alpha_1 - \alpha_2 \leq n + 1 - \alpha_1 \leq n$, we obtain that:
\[
f(n + 1)
\]
\[
\equiv f(n + 1 - \alpha_1) + f(n + 1 - \alpha_1 - \alpha_2) + \cdots + f(n + 1 - \alpha_1 - \alpha_2 - \cdots - \alpha_k) \pmod{m}
\]
\[
\equiv f(n + 1 - \alpha_1 + \alpha_2 + \cdots + \alpha_k - 1) \pmod{m}
\]
\[
\equiv f(n + 1 + l) \pmod{m}.
\]
It follows from the Principle of Strong Mathematical Induction that $f(n + l) \equiv f(n) \pmod{m}$ for $n \geq i$.

**Case 2.** Assume that $f(r + l) \equiv f(r) \pmod{m}$ for all $n \leq r \leq i + \alpha_1 + \alpha_2 + \cdots + \alpha_k - 1$ and $n \leq i$. Since $n \leq n + 1 - \alpha_1 - \alpha_2 - \cdots - \alpha_k \leq \cdots \leq n + 1 - \alpha_2 - \cdots - \alpha_k \leq \cdots \leq n - 1 + \alpha_2 + \cdots + \alpha_k \leq n - 1 + \alpha_1 + \alpha_2 + \cdots + \alpha_k \leq i$, we obtain that:
\[
f(n - 1)
\]
\[
\equiv f(n - 1 + \alpha_1 + \alpha_2 + \cdots + \alpha_k) - f(n - 1 + \alpha_2 + \cdots + \alpha_k) - \cdots - f(n - 1 + \alpha_k) \pmod{m}
\]
\[
\equiv f(n - 1 + \alpha_1 + \cdots + \alpha_k + l) - f(n - 1 + \alpha_2 + \cdots + \alpha_k + l) - \cdots - f(n - 1 + \alpha_k + l) \pmod{m}
\]
\[
\equiv f(n - 1 + l) \pmod{m}.
\]
It follows from the Principle of Strong Mathematical Induction that $f(n + l) \equiv f(n) \pmod{m}$ for $n \leq i$.

The proof is complete. \( \square \)

Theorem 2.3 tells us that there always exists a period of \((k: \alpha_1, \alpha_2, \ldots, \alpha_k)\)-step Fibonacci functions modulo $m$.

**Definition 2.4.** Let $f: \mathbb{Z} \to \mathbb{Z}$ be a \((k: \alpha_1, \alpha_2, \ldots, \alpha_k)\)-step Fibonacci function such that $f(n + l) \equiv f(n) \pmod{m}$ for any integer $n$ is called a **Period** of $f$ modulo $m$. The smallest positive integer $l$ such that $f(n + l) \equiv f(n) \pmod{m}$ for any integer $n$ is called the **Primitive Period** of $f$ modulo $m$ and write $l := l_f(m)$.

This unique primitive period always exists by The Well Ordering Principle (Burton, 2011). The following statements show some properties about a period and the primitive period of a \((k: \alpha_1, \alpha_2, \ldots, \alpha_k)\)-step Fibonacci function modulo $m$.

**Corollary 2.5.** (Tongron & Kerdmongkon, 2022) If $l_f(m)$ is the primitive period of a \((k: \alpha_1, \alpha_2, \ldots, \alpha_k)\)-step Fibonacci function $f$ modulo $m$, then $1 \leq l_f(m) \leq m^{\alpha_1 + \alpha_2 + \cdots + \alpha_k}$.

**Theorem 2.6.** (Tongron & Kerdmongkon, 2022) Let $f: \mathbb{Z} \to \mathbb{Z}$ be a \((k: \alpha_1, \alpha_2, \ldots, \alpha_k)\)-step Fibonacci function with $\alpha_k \geq 1$. For positive integers $l, m > 1, l$ is a period of $f$ modulo $m$ if and only if $l_f(m) | l$.

**Theorem 2.7.** (Tongron & Kerdmongkon, 2022) Let $f: \mathbb{Z} \to \mathbb{Z}$ be a \((k: \alpha_1, \alpha_2, \ldots, \alpha_k)\)-step Fibonacci function with $\alpha_k \geq 1$. If $gcd(m, n) = 1$, then $l_f(mn) = lcm\{l_f(m), l_f(n)\}$ for positive integers $m, n > 1$. 


**Example 2.8.** Let \( f: \mathbb{Z} \to \mathbb{Z} \) be a \((4; 1,0,0,1)\)-step Fibonacci function such that \( f(0) = 0 \) and \( f(1) = 1 \). Then we get the following tables:

<table>
<thead>
<tr>
<th>( n )</th>
<th>(-7)</th>
<th>(-6)</th>
<th>(-5)</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(n) )</td>
<td>1189</td>
<td>360</td>
<td>109</td>
<td>33</td>
<td>10</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>( f(n) \mod 2 )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( f(n) \mod 3 )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( f(n) \mod 6 )</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

We see that \( l_f(2) = 3 \leq 2^{1+0+0+1} \), \( l_f(3) = 2 \leq 2^{1+0+0+1} \), and \( l_f(6) = l_f(2 \cdot 3) = \text{lcm}[l_f(2), l_f(3)] = \text{lcm}[3,2] = 6 \leq 2^{1+0+0+1} \). Moreover, we observe that \( f(1) = f(-1) \), \( f(2) = -f(-2) \), \( f(3) = f(-3) \), \( f(4) = -f(-4) \) and so on. We can say that \( f \) is symmetric-like. This observation is explained in general as follows:

**Theorem 2.9.** Let \( f: \mathbb{Z} \to \mathbb{Z} \) be a \((k;1,0,\ldots,0,1)\)-step Fibonacci function with \( f(0) = 0 \). Then \( f(n) = (-1)^n+1 f(-n) \) for all non-negative integers \( n \).

**Proof.** It is obvious that \( f(0) = 0 = (-1)^0 f(-0) \) and \( f(1) = (k-1) f(0) + f(-1) = f(-1) = (-1)^2 f(-1) \). Assume that:

\[
f(t) = (-1)^t+1 f(-t)
\]

for all \( 1 \leq t \leq r \). Consider:

\[
f(r+1) = (k-1) f(r) + f(r-1)
\]
\[
= (k-1)[(-1)^r f(-r)] + (-1)^r f(-r + 1)
\]
\[
= (-1)^{r+2}[-(k-1) f(-r) + f(-r + 1)]
\]
\[
= (-1)^{r+2} f(-r - 1) .
\]

We are done.

We are motivated by Theorem 2.9 to ask if this fact for a \((k; a, 0,\ldots,0, a)\)-step Fibonacci function with \( a \geq 2 \) holds or not. First of all, let us consider the following example.

**Example 2.10.** Let \( f: \mathbb{Z} \to \mathbb{Z} \) be a \((5; 2,0,0,0,2)\)-step Fibonacci function such that \( f(0) = f(1) = f(2) = 0 \) and \( f(3) = -1 \). Then we get the following tables:

<table>
<thead>
<tr>
<th>( n )</th>
<th>(-9)</th>
<th>(-8)</th>
<th>(-7)</th>
<th>(-6)</th>
<th>(-5)</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(n) )</td>
<td>-305</td>
<td>0</td>
<td>72</td>
<td>0</td>
<td>-17</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( f(n) \mod 2 )</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-4</td>
<td>0</td>
<td>-17</td>
<td>0</td>
<td>-72</td>
</tr>
</tbody>
</table>

Again, we see that \( f \) is symmetric-like, i.e., \( f(2) = f(0) \), \( f(3) = f(-1) \), \( f(4) = f(-2) \), \( f(5) = -f(-3) \) and so on. To show that certain \((k; a, 0,\ldots,0, a)\)-step Fibonacci functions with \( a \geq 2 \) are symmetric-like, the following lemmas are important tools.

**Lemma 2.11.** Let \( a \) and \( a \) be integers with \( a \geq 2 \). Then the following statements are equivalent.

1. \( a \equiv a - 1 \mod 2a \) or \( a - a \equiv a - 1 \mod 2a \)
2. \( a \equiv a - 1 \mod a \)

**Proof.** It is clear that \( a \equiv a - 1 \mod 2a \) implies \( a \equiv a - 1 \mod a \). If \( a - a \equiv a - 1 \mod 2a \), then \( a - a \equiv a - 1 \mod a \) and so \( a \equiv a - 1 \mod a \). Conversely, assume that \( a \equiv a - 1 \mod a \) and \( a \equiv a - 1 \mod 2a \). It follows that:

\[
[a - (a - 1)]/a = a - (a - 1) \mod a - 1 \text{ is an odd integer.}
\]

Consequently, we obtain that:

\[
[(a - a) - (a - 1)]/a = [a - (a - 1)]/a - 1 \text{ is even.}
\]

In conclusion, \( a - a \equiv a - 1 \mod 2a \).

**Lemma 2.12.** Let \( f: \mathbb{Z} \to \mathbb{Z} \) be a \((k; a, 0,\ldots,0, a)\)-step Fibonacci function with \( a \geq 2 \) and \( f(n) = 0 \) for all \( 0 \leq n \leq 2a - 2 \). Then \( f(an + b) = 0 \) for all integers \( n \) and \( 0 \leq b \leq a - 2 \).

**Proof.** It is obvious that:

\[
f(a(1) + b) = 0 \quad \text{and} \quad f(a(2) + b) = (k - 1) f(a + b) + f(b) = 0
\]

because \( 0 < a + b \leq 2a - 2 \). Assume that \( r \in \mathbb{N} \) and
for all \(2 \leq t \leq r\). We get that:

\[
f(a(r + 1) + b) = (k - 1) f(a(r + 1) + b - a) + f(a(r + 1) + b - 2a) = (k - 1) f(ar + b) + f(a(r - 1) + b) = 0.
\]

In another direction, assume that \(r \in \mathbb{N}\) and

\[
f(at + b) = 0
\]

for all \(r \leq t \leq 1\). We get that:

\[
f(a(r - 1) + b) = -(k - 1) f(a(r - 1) + b + a) + f(a(r - 1) + 2a) = -(k - 1) f(ar + b) + f(a(r + 1) + b) = 0.
\]

This completes the proof by the Principle of Strong Mathematical Induction. \(\Box\)

Lemma 2.12 can be rewritten in a simple way as follows:

**Lemma 2.13.** Let \(f : \mathbb{Z} \rightarrow \mathbb{Z}\) be a \((k : a, 0, ..., 0, a)\) -step Fibonacci function with \(a \geq 2\) and \(f(n) = 0\) for all \(0 \leq n \leq 2a - 2\). If \(n \equiv a - 1 \pmod{a}\), then \(f(n) = 0\) for any integers \(n\).

We are now ready to prove the desired theorem.

**Theorem 2.14.** Let \(f : \mathbb{Z} \rightarrow \mathbb{Z}\) be a \((k : a, 0, ..., 0, a)\) -step Fibonacci function with \(a \geq 2\) and \(f(n) = 0\) for all \(0 \leq n \leq 2a - 2\). Then:

\[
f(n) = \begin{cases} f(-n + 2a - 2) & \text{if } n \not\equiv a - 1 \pmod{2a} \\ f(-n + 2a) & \text{if } n \equiv a - 1 \pmod{2a} \end{cases}
\]

for all \(n \geq a\).

**Proof.** It is not hard to see from the assumption that this statement holds for every \(a \leq n \leq 2a - 2\). Since:

\[
f(2a - 1) = (k - 1) f(a - 1) + f(-1) = f(-1) = f(-2a - 1 + 2a - 2)\]

and \(2a - 1 \not\equiv a - 1 \pmod{2a}\), this statement holds for \(n = 2a - 1\). Let \(n\) be an integer such that \(2a \leq n \leq 3a - 2\). Then

\[
f(n) = (k - 1) f(n - a) + f(n - 2a) = 0
\]

because \(0 \leq n - 2a < n - a \leq 2a - 2\). On the other hand, we obtain:

\[
f(-n + 2a - 2) = -(k - 1) f(-n - 2a) + f(-n - 2a + 3a) = 0
\]

because \(0 \leq -n - 2a < -n - a \leq 2a - 2\). Hence,

\[
f(n) = f(-n + 2a - 2)
\]

and we are done for this case since \(n \not\equiv a - 1 \pmod{2a}\). We observe from the above that:

\[
f(3a - 1) = (k - 1) f(2a - 1) + f(a - 1) = -(k - 1) f(-1) = f(a - 1) - f(a - 1) = -f(-3a - 1 + 2a - 2)
\]

and \(3a - 1 \equiv a - 1 \pmod{2a}\). Now the statement holds for all \(a \leq n \leq 3a - 1\). Let \(r \in \mathbb{N}\) and:

\[
f(t) = \begin{cases} f(-t + 2a - 2) & \text{if } t \not\equiv a - 1 \pmod{2a} \\ f(-t + 2a - 2) & \text{if } t \equiv a - 1 \pmod{2a} \end{cases}
\]

for all \(a \leq t \leq r\) and \(r \geq 3a - 1\). Note that:

\[
a \leq r + 1 - 2a < r + 1 - a < r.
\]

If \(r + 1 \equiv a - 1 \pmod{2a}\), then:

\[
r + 1 - a \equiv a - 1 \pmod{2a}
\]

and

\[
r + 1 - 2a \equiv a - 1 \pmod{2a}.
\]

These imply from the inductive assumption that:

\[
f(r + 1) = (k - 1) f(r + 1 - a) + f(r + 1 - 2a) = -f(-r + 1 + 2a - 2) + f(-r + 1 - 2a + 2a - 2) = -f(-r + 1 + 2a - 2).
\]

Next, assume that \(r + 1 \not\equiv a - 1 \pmod{2a}\). The proof is divided into 2 cases: \(r + 1 - a \equiv a - 1 \pmod{2a}\) and \(r + 1 - a \not\equiv a - 1 \pmod{2a}\).

**Case 1.** \(r + 1 - a \equiv a - 1 \pmod{2a}\). Then \(r + 1 - 2a \equiv a - 1 \pmod{2a}\). We get from the inductive assumption that:

\[
f(r + 1) = (k - 1) f(r + 1 - a) + f(r + 1 - 2a) = -(k - 1) f(-r + 1 + a + 2a - 2)
\]

and

\[
f(-r + 1 + 2a - 2).
\]

**Case 2.** \(r + 1 - a \not\equiv a - 1 \pmod{2a}\). By Lemma 2.11, we have \(r + 1 \not\equiv a - 1 \pmod{2a}\). We also have that \(-(r + 1) + 2a - 2 \equiv a - 1 \pmod{2a}\) since otherwise \(r + 1 \equiv a - 1 \pmod{2a}\): a contradiction. It follows from Lemma 2.13 that:

\[
f(r + 1) = 0 = f(-r + 1 + 2a - 2).
\]

The proof is complete by the Principle of Strong Mathematical Induction. \(\Box\)

Theorem 2.9 and Theorem 2.14 yield the next theorem.
Theorem 2.15. Let \( f: \mathbb{Z} \to \mathbb{Z} \) be a \((k; a, 0, \ldots, 0, a)\)-step Fibonacci function with \( a \in \mathbb{N} \) and \( f(n) = 0 \) for all \( 0 \leq n \leq 2a - 2 \). Then:

\[
  f(n) = \begin{cases} 
    f(-n + 2a - 2) & \text{if } n \not\equiv a - 1 \pmod{2a} \\
    -f(-n + 2a - 2) & \text{if } n \equiv a - 1 \pmod{2a}
  \end{cases}
\]

for all \( n \geq a \).

\[ \text{Proof.} \]

If \( a = 1 \), then we have from Theorem 2.9 that

\[ f(n) = (-1)^{n+1} f(-n) \]

for all non-negative integers \( n \). This shows that:

\[ f(n) = \begin{cases} 
    f(-n) & \text{if } n \not\equiv 0 \pmod{2} \\
    -f(-n) & \text{if } n \equiv 0 \pmod{2}
  \end{cases} \]

for all non-negative integers \( n \) and so the statement holds for \( a = 1 \). On the other hand, if \( a \geq 2 \), then the statement clearly holds from Theorem 2.14. \( \square \)

III. DISCUSSION AND CONCLUSION

In this paper, we have already defined \((k; a_1, a_2, \ldots, a_k)\)-step Fibonacci Functions and generalised Tongron and Kerdmongkon's work (Tongron & Kerdmongkon, 2022) which relates to periods of \( k \)-step Fibonacci Functions. It is also verified that some \((k; a_1, a_2, \ldots, a_k)\)-step Fibonacci Functions are symmetric-like as in Theorem 2.15. For the future work, we are going to provide some explicit formulae like Theorem 1.5, Theorem 1.6 and Theorem 1.8 for \((k; a_1, a_2, \ldots, a_k)\)-step Fibonacci Functions. Besides, we are inspired to establish a generalisation of Theorem 2.15 by the following examples: Let \( f: \mathbb{Z} \to \mathbb{Z} \) be a \((3; 1, 2, 1)\)-step Fibonacci function such that \( f(0) = 0, f(1) = 1, f(2) = -1 \) and \( f(3) = -2 \). Consider the following tables:

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(n) )</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
<td>-1</td>
<td>-4</td>
<td>-7</td>
<td>-9</td>
<td>-14</td>
</tr>
</tbody>
</table>

Observe that this \( f \) does not satisfy Theorem 2.15 but \( f \) seems symmetric-like.

IV. ACKNOWLEDGEMENT

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V. REFERENCES


Tongron, Y & Kerdmongkon, S 2022, 'Periods of \( k \)-step Fibonacci Functions Modulo \( m \)', Songklanakarin Journal of Science and Technology, vol. 44, no. 2, pp. 323-331.