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The Symmetries of $(k\colon \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci Functions

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It is well known that the Fibonacci sequence (F_n) is denoted by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, while the Lucas sequence (L_n) is denoted by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$. There are several studies showing relations between these two sequences. An interesting generalisation of both the sequences is a Fibonacci function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x+2) = f(x+1) + f(x)$ for any real number x (Elmore, 1967). Research about periods of Fibonacci numbers modulo m (Jameson, 2018) results in a contribution on the existence of primitive period of a Fibonacci function $f\colon \mathbb{Z} \to \mathbb{Z}$ modulo m (Thongngam & Chinram, 2019). Recently, a k-step Fibonacci function $f\colon \mathbb{Z} \to \mathbb{Z}$ denoted by $f(n+k) = f(n+k-1) + f(n+k-2) + \dots + f(n)$ for any integer n and $k \geq 2$ (which is a generalisation of a Fibonacci function $f\colon \mathbb{Z} \to \mathbb{Z}$) is introduced and the existence of primitive period of this function modulo m is established (Tongron & Kerdmongkon, 2022). In this work, let k be an integer ≥ 2. For nonnegative integers $\alpha_1, \alpha_2, \dots, \alpha_k$ and $\alpha_1 \neq 0$, a $(k\colon \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci function $f\colon \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = f(n-\alpha_1) + f(n-\alpha_1-\alpha_2) + \dots + f(n-\alpha_1-\alpha_2-\dots-\alpha_k)$ for any integer n. In fact, a k-step Fibonacci function is a special case of a $(k\colon \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci function. We present the existence of primitive period of this function modulo m and show that certain $(k\colon \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci functions are symmetric-like.

Keywords: Fibonacci functions; k-step Fibonacci function; $(k\colon \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci function; primitive period modulo m; symmetric-like

I. INTRODUCTION

The Fibonacci sequence (F_n) is defined by (Koshy, 2001; Vorob'ev, 2011):

$$F_0 = 0,\ F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2}$$

for any natural number $n \geq 2$. The beginning of the sequence is thus:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \dots$$

Similar to the Fibonacci sequence, the Lucas sequence (L_n) is defined by (Koshy, 2001):

$$L_0 = 2,\ L_1 = 1 \text{ and } L_n = L_{n-1} + L_{n-2}$$

for any natural number $n \geq 2$. The beginning of the sequence is thus:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, \dots$$

Recently, there are several interesting relations between the Fibonacci sequence and the Lucas sequence, for example, (Adegoke, 2022; Phunphayap, Khemaratchatakumthorn & Sumritnorrapong, 2022), etc.

In 1967, Elmore (Elmore, 1967) consider a relation between the Fibonacci sequence and the Lucas sequence and define a Fibonacci function $f\colon \mathbb{R} \to \mathbb{R}$ which is denoted by:

$$f(x+2) = f(x+1) + f(x)$$

for all real numbers x. Observe that if $f(0) = 0$ and $f(1) = 1$, then we get the Fibonacci sequence. Furthermore, if $f(0) = 2$ and $f(1) = 1$, then we get the Lucas sequence. Consequently, a Fibonacci function is a generalisation of both the Fibonacci sequence and the Lucas sequence.

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In 2018, Jameson (Jameson, 2018) studies periods of Fibonacci numbers modulo m and provides some properties on periods of such numbers. His work motivates Thongngam and Chinram (Thongngam & Chinram, 2019) to show the existence of primitive period of a Fibonacci function $f:\mathbb{Z}\to\mathbb{Z}$ modulo m. They also establish some relations among periods and the primitive periods of such functions.

Recently, Tongron and Kerdmongkon (Tongron & Kerdmongkon, 2022) study about a k-step Fibonacci function $f:\mathbb{Z}\to\mathbb{Z}$ defined by:

$$f(n+k)=f(n+k-1)+f(n+k-2)+\cdots+f(n)$$

for any integer n and $k\geq 2$. We can say equivalently that it is denoted by:

$$f(n)=f(n-1)+f(n-2)+\cdots+f(n-k)$$

for any integer n and $k\geq 2$. Observe that this function when $k=2$ is a generalisation of a Fibonacci function defined from $\mathbb{Z}$ to $\mathbb{Z}$. We refer to their work as follows:

Theorem 1.1. (Tongron & Kerdmongkon, 2022) *Let $f:\mathbb{Z}\to\mathbb{Z}$ be a k-step Fibonacci function and m be a positive integer >1. Then there exists an integer $1\leq l\leq m^k$ such that $f(n+l)\equiv f(n)\ (mod\ m)$ for any integer n.*

Such integer l is called a *Period* of f modulo m. If such integer l is the smallest, then it is called the *Primitive Period* of f modulo m and write $l\coloneqq l_f(m)$.

Theorem 1.2. (Tongron & Kerdmongkon, 2022) *Let $f:\mathbb{Z}\to\mathbb{Z}$ be a k-step Fibonacci function and l, m be positive integers >1. l is a period of f modulo m if and only if $l_f(m)\mid l$.*

Theorem 1.3. (Tongron & Kerdmongkon, 2022) *Let $f:\mathbb{Z}\to\mathbb{Z}$ be a k-step Fibonacci function and m, n be positive integers >1. If $gcd(m,n)=1$, then $l_f(mn)=lcm[l_f(m),l_f(n)]$.*

Indeed, Thongngam and Chinram's results (Thongngam & Chinram, 2019) are special cases of the above facts. Tongron and Kerdmongkon (Tongron & Kerdmongkon, 2022) also provide the explicit primitive periods of some k-step Fibonacci function as follows:

Lemma 1.4. (Tongron & Kerdmongkon, 2022) *Let m be a positive integer and $f:\mathbb{Z}\to\mathbb{Z}$ be a k-step Fibonacci function with the starting values $f(0)=a_0, f(1)=a_1, \ldots, f(k-1)=a_{k-1}$ and $gcd(m,k-1)=1$. Then $m|a_i$ for all $i\in\{0,1,\ldots,k-1\}$ if and only if $l_f(m)=1$.*

Theorem 1.5. (Tongron & Kerdmongkon, 2022) *Let $f:\mathbb{Z}\to\mathbb{Z}$ be a 2-step Fibonacci function with the starting values $f(0)=a$ and $f(1)=b$. Assume that $2m\nmid a$ or $2m\nmid b$. For a positive integer m, $m|a$ and $m|b$ if and only if $l_f(2m)=3$.*

Theorem 1.6. (Tongron & Kerdmongkon, 2022) *Let m be a positive odd integer and $f:\mathbb{Z}\to\mathbb{Z}$ be a 3-step Fibonacci function with the starting values $f(0)=a$, $f(1)=b$ and $f(2)=c$. Assume that $3m\nmid a$, $3m\nmid b$ or $3m\nmid c$. Then the following statements hold.*

(1) *If $m|a$, $m|b$ and $m|c$ then $l_f(3m)=13$.*

(2) *If $l_f(3m)=13$, then*

$$\begin{aligned}91a+141b+168c &\equiv 0\ (mod\ m)\\ 168a+259b+309c &\equiv 0\ (mod\ m)\\ 309a+477b+568c &\equiv 0\ (mod\ m).\end{aligned}$$

Corollary 1.7. (Tongron & Kerdmongkon, 2022) *Let $f:\mathbb{Z}\to\mathbb{Z}$ be a 3-step Fibonacci function with the starting values $f(0)=a$, $f(1)=b$ and $f(2)=c$. Then the following statements hold.*

(1) *If $l_f(9)=13$ and a, b or c is not divisible by 9, then $3|a$, $3|b$ and $3|c$.*

(2) *If $l_f(21)=13$ and a, b or c is not divisible by 21, then $7|a$, $7|b$ and $7|c$.*

Theorem 1.8. (Tongron & Kerdmongkon, 2022) *Let m be a positive integer and $f:\mathbb{Z}\to\mathbb{Z}$ be a 4-step Fibonacci function with the starting values $f(0)=a$, $f(1)=b$, $f(2)=c$ and $f(3)=d$. Assume that $gcd(4m,3)=1$ and a, b, c or d is not divisible by $4m$. Then the following statements hold.*

(1) *If $m|a$, $m|b$, $m|c$ and $m|d$, then*

$$l_f(4m)=\begin{cases}5 & \text{if } b+c+d, a+b+2c+2d, 2a+3b+3c+4d \\ & \text{and } 4a+6b+7c+7d \text{ are divisible by } 2m,\\ 10 & \text{otherwise.}\end{cases}$$

(2) *If $l_f(4m)=10$, then*

$$
\begin{aligned}
7a + 11b + 13c + 14d &\equiv 0 \ (mod\ m) \\
14a + 21b + 25c + 27d &\equiv 0 \ (mod\ m) \\
27a + 41b + 48c + 52d &\equiv 0 \ (mod\ m) \\
52a + 79b + 93c + 100d &\equiv 0 \ (mod\ m).
\end{aligned}
$$

In this paper, we define a $(k: \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci function $f: \mathbb{Z} \to \mathbb{Z}$ for nonnegative integers $\alpha_1, \alpha_2, \dots, \alpha_k$ and $\alpha_1 \neq 0$ by:

$$
f(n) = f(n - \alpha_1) + f(n - \alpha_1 - \alpha_2) + \cdots + f(n - \alpha_1 - \alpha_2 - \cdots - \alpha_k)
$$

for any integer n. Notice that the $(k: \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci function is a generalisation of a k-step Fibonacci function when all α are equal to 1. Theorem 1.1 – 1.3 are going to be proven in the version of $(k: \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci functions. There are also several examples to support our facts. Some of these examples motivate us to verify that certain $(k: \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci functions are symmetric-like.

II. MAIN RESULTS

Let k be an integer ≥ 2 and $\alpha_1, \alpha_2, \dots, \alpha_k$ be nonnegative integers such that $\alpha_1 \neq 0$. Recall that a $(k: \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci function $f: \mathbb{Z} \to \mathbb{Z}$ is defined by:

$$
f(n) = f(n - \alpha_1) + f(n - \alpha_1 - \alpha_2) + \cdots + f(n - \alpha_1 - \alpha_2 - \cdots - \alpha_k)
$$

for any integer n. For general use, a $(k: \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci function $f: \mathbb{Z} \to \mathbb{Z}$ satisfies:

$$
f(n) = f(n + \alpha_1 + \alpha_2 + \cdots + \alpha_k) - f(n + \alpha_2 + \cdots + \alpha_k) - f(n + \alpha_3 + \cdots + \alpha_k) - \cdots - f(n + \alpha_k)
$$

for any integer n.

Example 2.1. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a $(3: 2,1,1)$-step Fibonacci function such that $f(0) = 0$, $f(1) = 1$, $f(2) = -1$ and $f(3) = -2$. We can calculate the other $f(n)$ as follow:

$$
\begin{aligned}
&\vdots \\
f(-3) &= f(1) - f(-1) - f(-2) = 2 \\
f(-2) &= f(2) - f(0) - f(-1) = 2 \\
f(-1) &= f(3) - f(1) - f(0) = -3 \\
f(0) &= 0 \\
f(1) &= 1 \\
f(2) &= -1 \\
f(3) &= -2 \\
f(4) &= f(2) + f(1) + f(0) = 0 \\
f(5) &= f(3) + f(2) + f(1) = -2 \\
f(6) &= f(4) + f(3) + f(2) = -3 \\
&\vdots
\end{aligned}
$$

Then we get the following tables:

Table 1. The values of the $(3: 2,1,1)$-step Fibonacci function $f(n)$

n	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
$f(n)$	4	10	-7	-4	7	-1	-4	2	2	-3

n	0	1	2	3	4	5	6	7	8	9
$f(n)$	0	1	-1	-2	0	-2	-3	-4	-5	-9

Example 2.2. Let $f: \mathbb{Z} \to \mathbb{Z}$ be a $(4: 1,0,0,1)$-step Fibonacci function such that $f(0) = 0$ and $f(1) = 1$. We can calculate the other $f(n)$ as follow:

$$
\begin{aligned}
&\vdots \\
f(-3) &= f(-1) - f(-2) - f(-2) - f(-2) = 10 \\
f(-2) &= f(0) - f(-1) - f(-1) - f(-1) = -3 \\
f(-1) &= f(1) - f(0) - f(0) - f(0) = 1 \\
f(0) &= 0 \\
f(1) &= 1 \\
f(2) &= f(1) + f(1) + f(1) + f(0) = 3 \\
f(3) &= f(2) + f(2) + f(2) + f(1) = 10 \\
&\vdots
\end{aligned}
$$

Then we get the following tables:

Table 2. The values of the $(4: 1,0,0,1)$-step Fibonacci function $f(n)$

n	-8	-7	-6	-5	-4	-3	-2	-1
$f(n)$	-3927	1189	-360	109	-33	10	-3	1

n	0	1	2	3	4	5	6	7	8	9
$f(n)$	0	1	3	10	33	109	360	1189	3927	12970

Next, we show the existence of primitive period of $(k: \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci functions modulo m.

Theorem 2.3. *Let $f: \mathbb{Z} \to \mathbb{Z}$ be a $(k: \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci function with $\alpha_k \geq 1$ and m be a positive integer*

> 1. *Then there exists an integer* $1 \leq l \leq m^{\alpha_1+\alpha_2+\cdots+\alpha_k}$ *such that* $f(n+l) \equiv f(n) \pmod{m}$ *for any integer* n.

Proof. For any integer $a \in \{0,1,\dots,m^{\alpha_1+\alpha_2+\cdots+\alpha_k}\}$ which has $m^{\alpha_1+\alpha_2+\cdots+\alpha_k}+1$ elements, consider $(\alpha_1+\alpha_2+\cdots+\alpha_k)$-tuple $\big(f(a), f(a+1), \dots, f(a+\alpha_1+\alpha_2+\cdots+\alpha_k-1)\big)$ modulo m which can be $m^{\alpha_1+\alpha_2+\cdots+\alpha_k}$ possible values:

$$\begin{gathered}(0,0,\dots,0,0), (0,0,\dots,0,1), \dots, (0,0,\dots,0,m-1),\\ (0,0,\dots,1,0), (0,0,\dots,1,1), \dots, (0,0,\dots,1,m-1),\\ \vdots \qquad \vdots \qquad \vdots\\ (m-1,m-1,\dots,m-1,0), (m-1,m-1,\dots,m-1,1), \dots,\\ (m-1,m-1,\dots,m-1,m-1).\end{gathered}$$

We obtain from the Pigeonhole Principle (Burton, 2011) that there are integers $0 \leq i < j \leq m^{\alpha_1+\alpha_2+\cdots+\alpha_k}$ such that:

$$\begin{aligned}&\big(f(j), f(j+1), \dots, f(j+\alpha_1+\alpha_2+\cdots+\alpha_k-1)\big)\\ &\quad\equiv \big(f(i), f(i+1), \dots, f(i+\alpha_1+\alpha_2+\cdots+\alpha_k-1)\big) \pmod{m}.\end{aligned}$$

In other words, we have:

$$f(j+\alpha) \equiv f(i+\alpha) \pmod{m},$$

where $\alpha \in \{0,1,\dots,\alpha_1+\alpha_2+\cdots+\alpha_k-1\}$. Choose a positive integer $l \coloneqq j-i$ so that:

$$f(i+\alpha+l) \equiv f(i+\alpha) \pmod{m}$$

and $1 \leq l \leq m^{\alpha_1+\alpha_2+\cdots+\alpha_k}$. The proof is divided into two cases: $n \geq i$ and $n \leq i$.

Case 1. Assume that $f(r+l) \equiv f(r) \pmod{m}$ for $i \leq r \leq n$ and $n \geq i+\alpha_1+\alpha_2+\cdots+\alpha_k-1$. Since $i \leq n+1-\alpha_1-\alpha_2-\cdots-\alpha_k < n+1-\alpha_1-\alpha_2-\cdots-\alpha_{k-1} \leq \cdots \leq n+1-\alpha_1-\alpha_2 \leq n+1-\alpha_1 \leq n$, we obtain that:

$$\begin{aligned}&f(n+1)\\ &\equiv f(n+1-\alpha_1) + f(n+1-\alpha_1-\alpha_2) + \cdots\\ &\qquad + f(n+1-\alpha_1-\alpha_2-\cdots-\alpha_k) \pmod{m}\\ &\equiv f(n+1-\alpha_1+l) + f(n+1-\alpha_1-\alpha_2+l) + \cdots\\ &\qquad + f(n+1-\alpha_1-\alpha_2-\cdots-\alpha_k+l) \pmod{m}\\ &\equiv f(n+1+l) \pmod{m}.\end{aligned}$$

It follows from the Principle of Strong Mathematical Induction that $f(n+l) \equiv f(n) \pmod{m}$ for $n \geq i$.

Case 2. Assume that $f(r+l) \equiv f(r) \pmod{m}$ for all $n \leq r \leq i+\alpha_1+\alpha_2+\cdots+\alpha_k-1$ and $n \leq i$. Since $n \leq n-1+\alpha_k \leq \cdots \leq n-1+\alpha_2+\cdots+\alpha_k < n-1+\alpha_1+\alpha_2+\cdots+\alpha_k \leq i$, we obtain that:

$$\begin{aligned}&f(n-1)\\ &\equiv f(n-1+\alpha_1+\alpha_2+\cdots+\alpha_k) - f(n-1+\alpha_2+\cdots+\alpha_k)\\ &\qquad - \cdots - f(n-1+\alpha_k) \pmod{m}\\ &\equiv f(n-1+\alpha_1+\cdots+\alpha_k+l) - f(n-1+\alpha_2+\cdots+\alpha_k+l)\\ &\qquad - \cdots - f(n-1+\alpha_k+l) \pmod{m}\\ &\equiv f(n-1+l) \pmod{m}.\end{aligned}$$

It follows from the Principle of Strong Mathematical Induction that $f(n+l) \equiv f(n) \pmod{m}$ for $n \leq i$.

The proof is complete. □

Theorem 2.3 tells us that there always exists a period of $(k{:}\,\alpha_1,\alpha_2,\dots,\alpha_k)$-step Fibonacci functions modulo m.

Definition 2.4. *Let* $f: \mathbb{Z} \to \mathbb{Z}$ *be a* $(k{:}\,\alpha_1,\alpha_2,\dots,\alpha_k)$*-step Fibonacci function* $\alpha_k \geq 1$ *with and* m *be a positive integer* > 1. *A positive integer* l *such that* $f(n+l) \equiv f(n) \pmod{m}$ *for any integer* n *is called a* ***Period*** *of* f *modulo* m. *The smallest positive integer* l *such that* $f(n+l) \equiv f(n) \pmod{m}$ *for any integer* n *is called the* ***Primitive Period*** *of* f *modulo* m *and write* $l \coloneqq l_f(m)$.

This unique primitive period always exists by The Well Ordering Principle (Burton, 2011). The following statements show some properties about a period and the primitive period of a $(k{:}\,\alpha_1,\alpha_2,\dots,\alpha_k)$-step Fibonacci function modulo m. These facts can be verified similarly to Tongron and Kerdmongkon's work.

Corollary 2.5. (Tongron & Kerdmongkon, 2022) *If* $l_f(m)$ *is the primitive period of a* $(k{:}\,\alpha_1,\alpha_2,\dots,\alpha_k)$*-step Fibonacci function* f *modulo* m, *then* $1 \leq l_f(m) \leq m^{\alpha_1+\alpha_2+\cdots+\alpha_k}$.

Theorem 2.6. (Tongron & Kerdmongkon, 2022) *Let* $f: \mathbb{Z} \to \mathbb{Z}$ *be a* $(k{:}\,\alpha_1,\alpha_2,\dots,\alpha_k)$*-step Fibonacci function with* $\alpha_k \geq 1$. *For positive integers* $l, m > 1$, l *is a period of* f *modulo* m *if and only if* $l_f(m) \mid l$.

Theorem 2.7. (Tongron & Kerdmongkon, 2022) *Let* $f: \mathbb{Z} \to \mathbb{Z}$ *be a* $(k{:}\,\alpha_1,\alpha_2,\dots,\alpha_k)$*-step Fibonacci function with* $\alpha_k \geq 1$. *If* $gcd(m,n) = 1$, *then* $l_f(mn) = lcm\big[l_f(m), l_f(n)\big]$ *for positive integers* $m, n > 1$.

Example 2.8. Let $f:\mathbb{Z}\to\mathbb{Z}$ be a $(4:1,0,0,1)$-step Fibonacci function such that $f(0)=0$ and $f(1)=1$. Then we get the following tables:

Table 3. The values of the $(4:1,0,0,1)$-step Fibonacci function $f(n)$ in modulo 2, 3, and 6

n	-7	-6	-5	-4	-3	-2	-1
$f(n)$	1189	-360	109	-33	10	-3	1
$f(n)$ (mod 2)	1	0	1	1	0	1	1
$f(n)$ (mod 3)	1	0	1	0	1	0	1
$f(n)$ (mod 6)	1	0	1	3	4	3	1

n	0	1	2	3	4	5	6	7
$f(n)$	0	1	3	10	33	109	360	1189
$f(n)$ (mod 2)	0	1	1	0	1	1	0	1
$f(n)$ (mod 3)	0	1	0	1	0	1	0	1
$f(n)$ (mod 6)	0	1	3	4	3	1	0	1

We see that $l_f(2)=3\le 2^{1+0+0+1}$, $l_f(3)=2\le 3^{1+0+0+1}$ and $l_f(6)=l_f(2\cdot 3)=\mathrm{lcm}[l_f(2),l_f(3)]=\mathrm{lcm}[3,2]=6\le 6^{1+0+0+1}$. Moreover, we observe that $f(1)=f(-1)$, $f(2)=-f(-2)$, $f(3)=f(-3)$, $f(4)=-f(-4)$ and so on. We can say that f is symmetric-like. This observation is explained in general as follows:

Theorem 2.9. *Let $f:\mathbb{Z}\to\mathbb{Z}$ be a $(k:1,0,\dots,0,1)$-step Fibonacci function with $f(0)=0$. Then $f(n)=(-1)^{n+1}f(-n)$ for all non-negative integers n.*

Proof. It is obvious that $f(0)=0=(-1)^{0+1}f(-0)$ and

$$f(1)=(k-1)f(0)+f(-1)=f(-1)=(-1)^2f(-1).$$

Assume that:

$$f(t)=(-1)^{t+1}f(-t)$$

for all $1\le t\le r$. Consider:

$$\begin{aligned} f(r+1) &= (k-1)f(r)+f(r-1)\\ &= (k-1)[(-1)^{r+1}f(-r)]+(-1)^r f(-r+1)\\ &= (-1)^{r+2}[-(k-1)f(-r)+f(-r+1)]\\ &= (-1)^{r+2}f(-r-1). \end{aligned}$$

We are done. □

We are motivated by Theorem 2.9 to ask if this fact for a $(k:\alpha,0,\dots,0,\alpha)$-step Fibonacci function with $\alpha\ge 2$ holds or not. First of all, let us consider the following example.

Example 2.10. Let $f:\mathbb{Z}\to\mathbb{Z}$ be a $(5:2,0,0,0,2)$-step Fibonacci function such that $f(0)=f(1)=f(2)=0$ and $f(3)=-1$. Then we get the following tables:

Table 4. The values of the $(5:2,0,0,0,2)$-step Fibonacci function $f(n)$

n	-9	-8	-7	-6	-5	-4	-3	-2	-1
$f(n)$	-305	0	72	0	-17	0	4	0	-1

n	0	1	2	3	4	5	6	7	8	9	10	11
$f(n)$	0	0	0	-1	0	-4	0	-17	0	-72	0	-305

Again, we see that f is symmetric-like, i.e., $f(2)=f(0)$, $f(3)=f(-1)$, $f(4)=f(-2)$, $f(5)=-f(-3)$ and so on. To show that certain $(k:\alpha,0,\dots,0,\alpha)$-step Fibonacci functions with $\alpha\ge 2$ are symmetric-like, the following lemmas are important tools.

Lemma 2.11. *Let a and α be integers with $\alpha\ge 2$. Then the following statements are equivalent.*

(1) $a\equiv \alpha-1\ (mod\ 2\alpha)$ or $a-\alpha\equiv\alpha-1\ (mod\ 2\alpha)$

(2) $a\equiv\alpha-1\ (mod\ \alpha)$

Proof. It is clear that $a\equiv\alpha-1\ (\mathrm{mod}\ 2\alpha)$ implies $a\equiv\alpha-1\ (\mathrm{mod}\ \alpha)$. If $a-\alpha\equiv\alpha-1\ (\mathrm{mod}\ 2\alpha)$, then $a-\alpha\equiv\alpha-1\ (\mathrm{mod}\ \alpha)$ and so $a\equiv\alpha-1\ (\mathrm{mod}\ \alpha)$. Conversely, assume that $a\equiv\alpha-1\ (\mathrm{mod}\ \alpha)$ and $a\not\equiv\alpha-1\ (\mathrm{mod}\ 2\alpha)$. It follows that:

$$[a-(\alpha-1)]/\alpha \text{ is an odd integer.}$$

Consequently, we obtain that:

$$[(a-\alpha)-(\alpha-1)]/\alpha=[a-(\alpha-1)]/\alpha-1 \text{ is even.}$$

In conclusion, $a-\alpha\equiv\alpha-1\ (\mathrm{mod}\ 2\alpha)$. □

Lemma 2.12. *Let $f:\mathbb{Z}\to\mathbb{Z}$ be a $(k:\alpha,0,\dots,0,\alpha)$-step Fibonacci function with $\alpha\ge 2$ and $f(n)=0$ for all $0\le n\le 2\alpha-2$. Then $f(\alpha n+b)=0$ for all integers n and $0\le b\le\alpha-2$.*

Proof. It is obvious that:

$$f(\alpha(1)+b)=0 \text{ and}$$

$$f(\alpha(2)+b)=(k-1)f(\alpha+b)+f(b)=0$$

because $0<\alpha+b\le 2\alpha-2$. Assume that $r\in\mathbb{N}$ and

$$f(\alpha t+b)=0$$

for all $2 \leq t \leq r$. We get that:

$$\begin{aligned} f(\alpha(r+1)+b) &= (k-1)f(\alpha(r+1)+b-\alpha) \\ &\quad +f(\alpha(r+1)+b-2\alpha) \\ &= (k-1)f(\alpha r+b)+f(\alpha(r-1)+b) \\ &= 0. \end{aligned}$$

In another direction, assume that $r \in \mathbb{N}$ and

$$f(\alpha t+b)=0$$

for all $r \leq t \leq 1$. We get that:

$$\begin{aligned} f(\alpha(r-1)+b) &= -(k-1)f(\alpha(r-1)+b+\alpha) \\ &\quad +f(\alpha(r-1)+b+2\alpha) \\ &= -(k-1)f(\alpha r+b)+f(\alpha(r+1)+b) \\ &= 0. \end{aligned}$$

This completes the proof by the Principle of Strong Mathematical Induction. □

Lemma 2.12 can be rewritten in a simple way as follows:

Lemma 2.13. *Let $f:\mathbb{Z}\to\mathbb{Z}$ be a $(k:\alpha,0,\dots,0,\alpha)$-step Fibonacci function with $\alpha \geq 2$ and $f(n)=0$ for all $0 \leq n \leq 2\alpha-2$. If $n \not\equiv \alpha-1 \ (mod\ \alpha)$, then $f(n)=0$ for any integers n.*

We are now ready to prove the desired theorem.

Theorem 2.14. *Let $f:\mathbb{Z}\to\mathbb{Z}$ be a $(k:\alpha,0,\dots,0,\alpha)$-step Fibonacci function with $\alpha \geq 2$ and $f(n)=0$ for all $0 \leq n \leq 2\alpha-2$. Then:*

$$f(n)=\begin{cases} f(-n+2\alpha-2) & if\ n \not\equiv \alpha-1\ (mod\ 2\alpha) \\ -f(-n+2\alpha-2) & if\ n \equiv \alpha-1\ (mod\ 2\alpha) \end{cases}$$

for all $n \geq \alpha$.

Proof. It is not hard to see from the assumption that this statement holds for every $\alpha \leq n \leq 2\alpha-2$. Since:

$$\begin{aligned} f(2\alpha-1) &= (k-1)f(\alpha-1)+f(-1) \\ &= f(-1) \\ &= f(-(2\alpha-1)+2\alpha-2) \end{aligned}$$

and $2\alpha-1 \not\equiv \alpha-1 \ (\text{mod } 2\alpha)$, this statement holds for $n=2\alpha-1$. Let n be an integer such that $2\alpha \leq n \leq 3\alpha-2$. Then

$$f(n)=(k-1)f(n-\alpha)+f(n-2\alpha)=0$$

because $0 \leq n-2\alpha < n-\alpha \leq 2\alpha-2$. On the other hand, we obtain:

$$f(-n+2\alpha-2)=-(k-1)f(-n-2+3\alpha)$$
$$+f(-n-2+4\alpha)=0$$

because $0 \leq -n-2+3\alpha < -n-2+4\alpha \leq 2\alpha-2$. Hence,

$$f(n)=f(-n+2\alpha-2)$$

and we are done for this case since $n \not\equiv \alpha-1 \ (\text{mod } 2\alpha)$. We observe from the above that:

$$\begin{aligned} f(3\alpha-1) &= (k-1)f(2\alpha-1)+f(\alpha-1) \\ &= -[-(k-1)f(-1)] \\ &= -[f(-\alpha-1)-f(\alpha-1)] \\ &= -f(-(3\alpha-1)+2\alpha-2) \end{aligned}$$

and $3\alpha-1 \equiv \alpha-1 \ (\text{mod } 2\alpha)$. Now the statement holds for all $\alpha \leq n \leq 3\alpha-1$. Let $r \in \mathbb{N}$ and:

$$f(t)=\begin{cases} f(-t+2\alpha-2) & \text{if } t \not\equiv \alpha-1 \ (\text{mod } 2\alpha) \\ -f(-t+2\alpha-2) & \text{if } t \equiv \alpha-1 \ (\text{mod } 2\alpha) \end{cases}$$

for all $\alpha \leq t \leq r$ and $r \geq 3\alpha-1$. Note that:

$$\alpha \leq r+1-2\alpha < r+1-\alpha < r.$$

If $r+1 \equiv \alpha-1 \ (\text{mod } 2\alpha)$, then:

$$r+1-\alpha \not\equiv \alpha-1 \ (\text{mod } 2\alpha) \text{ and}$$
$$r+1-2\alpha \equiv \alpha-1 \ (\text{mod } 2\alpha).$$

These imply from the inductive assumption that:

$$\begin{aligned} & f(r+1) \\ &= (k-1)f(r+1-\alpha)+f(r+1-2\alpha) \\ &= -[-(k-1)f(-(r+1-\alpha)+2\alpha-2) \\ &\qquad +f(-(r+1-2\alpha)+2\alpha-2)] \\ &= -f(-(r+1)+2\alpha-2). \end{aligned}$$

Next, assume that $r+1 \not\equiv \alpha-1 \ (\text{mod } 2\alpha)$. The proof is divided into 2 cases: $r+1-\alpha \equiv \alpha-1 \ (\text{mod } 2\alpha)$ and $r+1-\alpha \not\equiv \alpha-1 \ (\text{mod } 2\alpha)$.

Case 1. $r+1-\alpha \equiv \alpha-1 \ (\text{mod } 2\alpha)$. Then $r+1-2\alpha \not\equiv \alpha-1 \ (\text{mod } 2\alpha)$. We get from the inductive assumption that:

$$\begin{aligned} & f(r+1) \\ &= (k-1)f(r+1-\alpha)+f(r+1-2\alpha) \\ &= -(k-1)f(-(r+1-\alpha)+2\alpha-2) \\ &\qquad +f(-(r+1-2\alpha)+2\alpha-2) \\ &= f(-(r+1)+2\alpha-2). \end{aligned}$$

Case 2. $r+1-\alpha \not\equiv \alpha-1 \ (\text{mod } 2\alpha)$. By Lemma 2.11, we have $r+1 \not\equiv \alpha-1 \ (\text{mod } \alpha)$. We also have that $-(r+1)+2\alpha-2 \not\equiv \alpha-1 \ (\text{mod } \alpha)$ since otherwise $r+1 \equiv \alpha-1 \ (\text{mod } \alpha)$: a contradiction. It follows from Lemma 2.13 that:

$$f(r+1)=0=f(-(r+1)+2\alpha-2).$$

The proof is complete by the Principle of Strong Mathematical Induction. □

Theorem 2.9 and Theorem 2.14 yield the next theorem.

Theorem 2.15. *Let* $f: \mathbb{Z} \to \mathbb{Z}$ *be a* $(k: \alpha, 0, \dots, 0, \alpha)$ *-step Fibonacci function with* $\alpha \in \mathbb{N}$ *and* $f(n) = 0$ *for all* $0 \le n \le 2\alpha - 2$. *Then:*

$$f(n) = \begin{cases} f(-n + 2\alpha - 2) & if\ n \not\equiv \alpha - 1 \pmod{2\alpha} \\ -f(-n + 2\alpha - 2) & if\ n \equiv \alpha - 1 \pmod{2\alpha} \end{cases}$$

for all $n \ge \alpha$.

Proof. If $\alpha = 1$, then we have from Theorem 2.9 that

$$f(n) = (-1)^{n+1}\ f(-n)$$

for all non-negative integers n. This shows that:

$$f(n) = \begin{cases} f(-n) & if\ n \not\equiv 0 \pmod 2 \\ -f(-n) & if\ n \equiv 0 \pmod 2 \end{cases}$$

for all non-negative integers n and so the statement holds for $\alpha = 1$. On the other hand, if $\alpha \ge 2$, then the statement clearly holds from Theorem 2.14. □

III. DISCUSSION AND CONCLUSION

In this paper, we have already defined $(k: \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci Functions and generalised Tongron and Kerdmongkon's work (Tongron & Kerdmongkon, 2022) which relates to periods of k-step Fibonacci Functions. It is also verified that some $(k: \alpha_1, \alpha_2, \dots, \alpha_k)$ -step Fibonacci Functions are symmetric-like as in Theorem 2.15. For the future work, we are going to provide some explicit formulae like Theorem 1.5, Theorem 1.6 and Theorem 1.8 for $(k: \alpha_1, \alpha_2, \dots, \alpha_k)$-step Fibonacci Functions. Besides, we are inspired to establish a generalisation of Theorem 2.15 by the following examples: Let $f: \mathbb{Z} \to \mathbb{Z}$ be a $(3: 1,2,1)$ -step Fibonacci function such that $f(0) = 0$, $f(1) = 1$, $f(2) = -1$ and $f(3) = -2$. Consider the following tables:

Table 5. The values of the $(3: 1,2,1)$-step Fibonacci function $f(n)$

n	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
$f(n)$	-25	14	-9	7	-4	1	-1	2	-1	-1

n	0	1	2	3	4	5	6	7	8	9
$f(n)$	0	1	-1	-2	-1	-1	-4	-7	-9	-14

Observe that this f does not satisfy Theorem 2.15 but f seems symmetric-like.

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