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Computation of Neumann Localised Boundary Domain Integral Equations

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Most integrals in Localised Boundary Domain Integral Equations (LBDIEs) comprise singularities. This paper aims to produce numerical solutions of the LBDIEs for the Partial Differential Equations with variable coefficients. The singularities of the boundary integrals in LBDIEs will be handled by using a semi-analytic for logarithmic singularity and a semi-quadratic analytic method for r^{-2} singularity. Whereas the singular domain integrals are handled by using the Duffy transformation. The LBDIEs that we consider are associated with the Neumann problem, which can be solved with a condition. If it can be solved, the solution is, however, unique up to an additive constant. We add a perturbation operator to the LBDIEs to convert the LBDIE to a uniquely solvable equation. The perturbed integral operator leads the perturbed LBDIEs to a dense matrix system that disable the use of methods in solving sparse matrix system. We solve the system of linear equations by Lower-Upper (LU) decomposition method. The numerical results indicate that high accuracy results can be attained. It gives the impression that the methods we use in this numerical experiment are reliable in handling the boundary and domain singular integrals.

Keywords: boundary element method; localised boundary domain integral equations; Neumann problem; partial differential equations; variable coefficient

I. INTRODUCTION

The Boundary Element Model (BEM) is a method for the numerical solution of partial differential equations used in engineering problems. There are many other methods for solving partial differential equations. See e.g. (Melenk & Xenophontos, 2015; Melenk *et al.*, 2012; Nolasco *et al.*, 2020; Brandenburg & Clemmons, 2012) for the discussion of other methods for instance, finite-difference method (FDM), hp-Finite Element Method and Finite Element Method (FEM). Examples of BEM's applications can be attained from the fields of elastodynamics, electromagnetics, acoustics, biomechanics, and off-shore structures, which can be found in (Chaillat *et al.*, 2017; Xu *et al.*, 2018; Wang *et al.*, 2012; Kirkup, 2019; Katsikadelis, 2016).

The BEM is established through the transformation of the differential equation into an integral equation that is well-grounded throughout the domain, on the boundary, and outside the domain. Some examples of differential equations include Laplace's equation, Helmholtz's equation, the convection-diffusion equation, the potential equation, the equation of viscous flow, equations of electrostatics and electromagnetics, and the equations of linear elastostatics and elastodynamics as explained in (Costaz, 2002).

In most cases in BEM, the unknown function or its derivative is solved only for its boundary distribution. The

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solution at each internal domain point can be generated by direct assessment after the unknown boundary distributions are determined. Other methods, e.g., finite differences and finite element methods, require the whole domain to be discretised, and this greatly increases the computational cost and time consumption. See e.g. (Brandenburg & Clemmons, 2012; Sutradhar *et al.*, 2008).

In this paper, we will consider an elliptic partial differential equation with a variable coefficient with 2 dimensions. The Neumann boundary condition is prescribed on the boundary. Unlike Laplace's equation which transforms into a boundary integral equation, an elliptic partial differential equation with a variable coefficient will be transformed into Boundary-Domain Integral Equation.

The required function must, therefore, be determined throughout the entire boundary and the interior region of the solution. The boundary integral equations that include the fundamental solution and its first derivative are obtained for the Laplace equations corresponding to an elliptical equation with a constant coefficient. Nonetheless, the normal derivative of the fundamental solution is also a solution for the differential equation.

For the elliptic equation with a variable coefficient, the fundamental solution is generally unknown. However, a parametrix is generally available, enabling the derivation of the boundary-domain integral equations for potential and flux. See, e.g. (Mikhailov & Mohamed, 2012; Mohamed *et al.*, 2016).

A negative aspect of the boundary integral equation and Boundary-Domain Integral Equations is that the resulting discrete matrix equations are dense matrices. See, e.g. (Mohamed *et al.*, 2016). Finite elements and finite differences methods produce matrix systems that are sparse and inherently greater in size than the matrix systems generated by boundary integral equations. See, e.g. (Nolasco *et al.*, 2020; Brandenburg & Clemmons, 2012). However, the system of matrix produced by finite elements and finite differences methods and Boundary-Domain Integral Equations are of the same size.

Unlike FEM, Boundary-Domain Integral Equations (BDIEs) comprise singular integrals of logarithmic and its derivative kernels. Whenever the source point and field point overlap, its derivative's singularity is increasingly strengthened. It is well-known that the computation of extremely singular integral is crucial.

The sizes of matrices produced by finite elements, finite differences methods, and Boundary-Domain Integral Equations are of the same size. And yet, the disadvantage of having singularity kernels like in Boundary Integral Equations (BIEs) preserve in BDIEs cases. Furthermore, the discrete matrix is a dense matrix which is again a drawback as compared to FDM and FEM.

Therefore, (Mohamed *et al.*, 2016) introduced a localised parametric to transform a partial differential equation with a variable coefficient into a Localised Boundary integral equation. The matrix systems generated by LBDIEs are sparse, as in FEM and FDM. However, the drawback of having singularity kernels like in BIEs and BDIEs still preserves in LBDIEs cases.

Purely Neumann problem can be solved with a condition. If it can be solved, the solution is, however, unique up to an additive constant. See, e.g. (Mikhailov & Mohamed, 2012). Neumann LBDIDE inherits this property. In this paper, we append a perturbation operator to the LBDIEs to convert the LBDIE to a uniquely solvable equation.

Boundary integral is an emerging area of research, and recent developments have been made in several new computational methods. Some methods in recasting the integral equations to a system of equations are, for instance, the collocation method, Galerkin, Boundary Contour Method, Boundary Node Method, and Boundary Cloud Method, as mentioned in (Sutradhar *et al.*, 2008).

The first mentioned method, i.e., the collocation method, can also be utilised for the BDIEs and LBDIEs. In this collocation method, the boundary integral equation is expressed at a number of points. See, e.g. (Katsikadelis, 2016; Beer *et al.*, 2001). The simplest collocation method is by taking the nodes in discretising the boundary as the collocation points. Some works on the BDIEs related to the collocation method can be found in (Mikhailov & Mohamed, 2012; Mohamed *et al.*, 2016a; Mohamed *et al.*, 2016b).

In this paper, we will use the collocation point method with linear interpolation and bilinear interpolation for boundary and domain integration, respectively. The numerical integration for each of the elements is accomplished with

Gauss-Legendre quadrature, except for those elements in which the integration is singular.

The BIEs, BDIEs, and LBDIEs consist of singular integrals of logarithmic and its derivative kernels. These singular integrals diverge whenever source and field points overlap. The singular integration over elements for integrals of logarithmic can be performed by the Gauss-Laguerre quadrature formula. Some numerical results that use Gauss-Laguerre quadrature formula for the singular logarithmic integrals are e.g. (Mikhailov & Mohamed, 2012; Beer *et al.*, 2001).

Mohamed (2014) introduced a semi-analytic method that is an alternative to the Gauss-Laguerre quadrature formula for boundary singular integration over elements of logarithmic integrals. This semi-analytic method was then utilised for solving numerical BDIEs/BDIDEs. See e.g. (Mohamed *et al.*, 2016a; Mohamed *et al.*, 2016b). Mohamed *et al.* (2020) derived a method named as semi-quadratic analytic method, which is a method to handle boundary integration with r^{-2} singularity.

In this paper, we will solve numerical LBDIE related to the Neumann problem. We utilise the semi-analytic method formula for boundary singular logarithmic integration. In addition, we use semi-quadratic analytic method for boundary singular integration with r^{-2} singularity. The mesh-based discretisation of the unperturbed Neumann LBDIEs by using quadrilateral domain elements leads to the system of linear equations with the sparse matrix operator. However, the perturbed integral operator gives a dense matrix that leads the perturbed LBDIEs to a dense matrix system that disables the use of methods for solving sparse matrix system.

II. MATERIALS AND METHOD

A. Localised Boundary-Domain Integral Equations

A linear second-order elliptic PDE with a variable coefficient $a(\xi)$ is taken into account, as given below.

$$Au(\xi)=\sum_{i=1}^{n}\frac{\partial}{\partial\xi_i}a(\xi)\frac{\partial}{\partial\xi_i}u(\xi)=f(\xi), \tag{1}$$

where $u(\xi)$ is an undefined function, whereas $f(\xi)$ is known.

Mikhailov (2002) discussed the use of a localised parametrix $Q_\chi(\xi,\eta)$ that can transform partial differential equations with variable coefficient $a(\xi)$ in (1) to a sparse matrix system.

The localised parametrix $Q_\chi(\xi,\eta)$ is given as follows:

$$Q_\chi(\xi,\eta)=\chi(\xi,\eta)Q(\xi,\eta),$$

where $Q(\xi,\eta)$ is a parametrix given by

$$Q(\xi,\eta)=\frac{ln\,|\xi-\eta|}{2\pi a(\eta)},\ \ \xi,\eta\in\mathbb{R}^2,$$

and $\chi(\xi,\eta)$ is a discontinuation function. It implies that $\chi(\eta,\eta)=1$ and $\chi(\xi,\eta)=0$ at η not part of a localisation domain $\omega(\eta)$.

Therefore, $Q_\chi(\xi,\eta)$ is not zero only on $\omega(\eta)$. The singularity of $Q_\chi(\xi,\eta)$ is inherited from $Q(\xi,\eta)$ whenever the overlapping between source and field points occurs.

The localised parametrix $Q_\chi(\xi,\eta)$ satisfies that

$$A_\xi Q_\chi(\xi,\eta)=\delta(\xi,\eta)+S_\chi(\xi,\eta).$$

Here $\delta(\xi,\eta)$ is the Dirac delta function, and the localised remainder $S_\chi(\xi,\eta)$ is denoted as

$$S_\chi(\xi,\eta)=S(\xi,\eta)-A_\xi((1-\chi)Q),$$

where,

$$S(\xi,\eta)=\frac{1}{2\pi a(\eta)}\sum_{i=1}^{2}\frac{\xi_i-\eta_i}{r}\frac{\partial a(\xi)}{\partial\xi_i},\ \ \xi,\eta\in\mathbb{R}^2.$$

The third Green identity related to localised parametrix $Q_\chi(\xi,\eta)$ is as follows. See e.g. (Mikhalov, 2002).

$$\begin{aligned}
&c(\eta)u(\eta)-\int_{\overline{w}(y)\cap\partial\Omega}T_\xi Q_\chi(\xi,\eta)u(\xi)\,\mathrm{d}\Gamma(\xi)\\
&+\int_{\overline{w}(\eta)\cap\partial\Omega}Tu(\xi)Q_\chi(\xi,\eta)\,\mathrm{d}\Gamma(\xi)\\
&-\int_{\Omega\cap\partial\omega(\eta)}T_\xi Q_\chi(x,\eta)u(\xi)\,\mathrm{d}\Gamma(\xi)\\
&+\int_{\Omega\cap\partial\omega(\eta)}Tu(\xi)Q_\chi(\xi,\eta)\,\mathrm{d}\Gamma(\xi)\\
&+\int_{w(\eta)\cap\Omega}u(\xi)S_\chi(\xi,\eta)\,\mathrm{d}\Omega(\xi)\\
&=\int_{w(\eta)\cap\Omega}f(\xi)Q_\chi(\xi,\eta)\,\mathrm{d}\Omega(\xi),\quad \eta\in\overline{\Omega}.
\end{aligned} \tag{2}$$

where,

$$T_\xi Q(\xi,\eta)=\sum_{i=1}^{2}a(\xi)\nu_i(\xi)\frac{\partial Q(\xi,\eta)}{\partial \xi_i}$$
$$=\sum_{i=1}^{2}a(\xi)\nu_i(\xi)\frac{(\xi_i-\eta_i)}{2\pi a(\eta)r^2}.$$

Here Ω is the domain and $\partial\Omega$ is the boundary such that $\Omega\cup\partial\Omega=\bar{\Omega}$ and $\Omega\cap\partial\Omega=\varnothing$.

In addition, we define the localisation domain and localisation boundary as ω and $\partial\omega$, respectively, with $\omega\cup\partial\omega=\bar{\omega}$ and $\omega\cap\partial\omega=\varnothing$.

The discontinuation function $\chi(\xi,y)$ is taken as

$$\chi(\xi,\eta)=\begin{cases}1, & \xi\in\omega(\eta),\\ 0, & \xi\notin\omega(\eta),\end{cases}$$

that leads

$$Q_\chi(\xi,\eta)=\begin{cases}Q(\xi,\eta), & \xi\in\omega(\eta),\\ 0, & \xi\notin\omega(\eta),\end{cases}$$

and

$$Q_\chi(\xi,\eta)=\begin{cases}Q(\xi,\eta), & \xi\in\omega(\eta),\\ 0, & \xi\notin\omega(\eta).\end{cases}$$

As compared to BDIEs, the LBDIEs has additional two integrals due to the fact that $\chi(\xi,\eta)$ is discontinuance in $\xi\in\mathbb{R}^n$.

B. *Localised Boundary-Domain Integral Equation Method for Neumann Problem*

For Neumann problem, we apply the Neumann boundary condition $Tu(\xi)=\bar{t}(\xi),\ \xi\in\partial\Omega$ in (2), gives

$$\begin{aligned}&c(\eta)u(\eta)-\int_{\partial\omega(y)}T_\xi Q_\chi(\xi,\eta)u(\xi)\,\mathrm{d}\Gamma(\xi)\\&+\int_{\Omega\cap\partial\omega(\eta)}Tu(\xi)Q_\chi(\xi,\eta)\,\mathrm{d}\Gamma(\xi)\\&+\int_{\omega(\eta)\cap\Omega}u(\xi)S_\chi(\xi,\eta)\,\mathrm{d}\Omega(\xi)\\&=-\int_{\bar{\varpi}(\eta)\cap\partial\Omega}\bar{t}(\xi)Q_\chi(\xi,\eta)\,\mathrm{d}\Gamma(\xi)\\&+\int_{\omega(\eta)\cap\Omega}f(\xi)Q_\chi(\xi,\eta)\,\mathrm{d}\Omega(\xi),\quad \eta\in\bar{\Omega}.\end{aligned}\tag{3}$$

The solution u is not determined uniquely, but to the approximation of an arbitrary constant. The uniqueness can be attained by adding the perturbation operator (4),

$$\frac{1}{|\partial\Omega|}\int_{\partial\Omega}u(\xi)\,\mathrm{d}\Gamma(\xi),\tag{4}$$

to the Neumann LBDIE (3) and yields as described in the following:

$$\begin{aligned}&c(\eta)u(\eta)+\frac{1}{|\partial\Omega|}\int_{\partial\Omega}u(\xi)\,\mathrm{d}\Gamma(\xi)\\&-\int_{\bar{\varpi}(\eta)}T_\xi Q_\chi(\xi,\eta)u(\xi)\,\mathrm{d}\Gamma(\xi)\\&+\int_{\Omega\cap\partial\omega(\eta)}Tu(\xi)Q_\chi(\xi,\eta)\,\mathrm{d}\Gamma(\xi)\\&+\int_{\omega(\eta)\cap\Omega}u(\xi)S_\chi(\xi,\eta)\,\mathrm{d}\Omega(\xi)\\&=-\int_{\bar{\varpi}(\eta)\cap\partial\Omega}\bar{t}(\xi)Q_\chi(\xi,\eta)\,\mathrm{d}\Gamma(\xi)\\&+\int_{\omega(\eta)\cap\Omega}f(\xi)Q_\chi(\xi,\eta)\,\mathrm{d}\Omega(\xi),\quad \eta\in\bar{\Omega}.\end{aligned}\tag{5}$$

The adding of the perturbation operator (4) is also used in (Mikhailov & Mohamed, 2012) for solving Neumann BDIEs. We use collocation method to reduce LBDIE to the finite system. In this method, the LBDIEs are implemented at a number of points on the boundary as well as interior domain. The nodes that are used to discretise the boundary and mesh the interior domain are then taken as the collocation points. Note that, from (5), the localised parametrix $Q_\chi(\xi,\eta)$ kernels in the 3rd, 5th and 6th integrals are generally die off for large $r=|\xi-\eta|$. Therefore, when the value of $r=|\xi-\eta|$ is close to zero i.e., whenever $\xi\in\omega(\eta)$, the singular integrals incline to be the prevailing terms.

The same case happened to its derivative $T_\xi Q_\chi(\xi,\eta)$ kernel in the 2nd integral and the localised remainder kernel $S_\chi(\xi,\eta)$ in 4th integral. Therefore, excellent methods to handle the singularities integrations of the localised parametrix $Q_\chi(\xi,\eta)$, its derivative $T_\xi Q_\chi(\xi,\eta)$ and localised remainder $S_\chi(\xi,\eta)$ are necessary.

We chose Duffy transformation to handle all the domain integrations with the singularity kernels, i.e., for the 4th and 6th integrals. For the boundary integrations with the singularity kernels, we used 3 different methods which are most appropriate for each of the integral. See (Duffy, 1982) for the discussion on Duffy transformation.

We use semi-analytic method and semi-quadratic analytic method as proposed by (Mohamed, 2014) and (Mohamed *et al.*, 2020), respectively to handle the boundary singularity in the 3rd integral and 2nd integral before the integrals are calculated by the standard Gaussian quadrature method i.e., Gauss-Legendre quadrature formula. The singularity boundary integral in the 5th integral is calculated by Gauss-Laguerre quadrature formula.

III. RESULT AND DISCUSSION

In this section, we demonstrate the numerical solution for perturbed Neumann LBDIE (5) for three test domains. We apply exactly identical test domains like in (Mikhailov & Mohamed, 2012; Mohamed *et al.*, 2016), i.e., a square, a circle and a parallelogram for easier comparability.

Below is the list of the Neumann problems that we use as our test problems.

Test 1:

$$\begin{aligned} &a(\xi) = \xi_2^2,\ f(\xi) = 0\ for\ \xi \in \Omega \cup \partial\Omega, \\ &\text{with } t(\xi) = \xi_2^2\ \nu_1(\xi),\ \xi \in \partial\Omega, \end{aligned} \tag{6}$$

Test 2:

$$\begin{aligned} &a(\xi) = \xi_2^2,\ f(\xi) = 2\xi_2^2\ for\ \xi \in \Omega \cup \partial\Omega, \\ &\text{with } t(\xi) = 2\xi_1\xi_2\ 2\nu_1(\xi),\ \xi \in \partial\Omega, \end{aligned} \tag{7}$$

The problems in tests 1-2 in (6) and (7) have the exact solutions as in (8) and (9), respectively.

$$u(\xi) = \xi_1,\ \xi \in \Omega \cup \partial\Omega, \tag{8}$$

$$u(\xi) = \xi_1^2,\ \xi \in \Omega \cup \partial\Omega. \tag{9}$$

The Neumann problems as in (6) and (7) are the same problems as tested in (Mikhailov & Mohamed, 2012) but now solved for LBDIEs.

The numerical implementations are performed by employing Fortran up to double precision accuracy. The solution of the matrix system yields from perturbed Neumann LBDIE (5) is attained by using LU decomposition method.

We express relative errors for the estimated solution and its gradient for each of the domains as given below.

$$\epsilon(u) = \frac{\max\limits_{1\le j\le J}\left|u_{approx}(\xi^j) - u_{exact}(\xi^j)\right|}{\max\limits_{1\le j\le J}\left|u_{exact}(\xi^j)\right|}, \tag{10}$$

$$\epsilon(\nabla u) = \frac{\max\limits_{1\le m\le M}\left|\nabla u_{approx}(\xi_c^m) - \nabla u_{exact}(\xi_c^m)\right|}{\max\limits_{1\le m\le M}\left|\nabla u_{exact}(\xi_c^m)\right|}, \tag{11}$$

where ξ_c^m centres quadrilateral domain elements e_m.

The following are the tables for computational relative errors of the approximate solution $\epsilon(u)$ and of its gradient $\epsilon(\nabla u)$ as given in (10) and (11). Tables 1-3 display the relative errors of approximate solutions u_{approx} and their gradient ∇u_{approx} versus number of nodes J for Test 1 and Test 2.

Table 1. Relative errors on the square vs. number of nodes J of the estimated solutions (a) and their gradients (b).

J	$\epsilon(u)$ **for Test 1**	$\epsilon(u)$ **for Test 2**	$\epsilon(\nabla u)$ **for Test 1**	$\epsilon(\nabla u)$ **for Test 2**
25	2.211E-10	1.470E-2	1.242E-09	8.632E-2
81	1.193E-10	1.086E-2	1.079E-09	6.605E-2
289	6.946E-11	5.586E-3	9.48E-10	4.246E-2
1089	4.371E-11	2.682E-3	8.447E-10	2.402E-2

Table 2. Relative errors on the circular domain vs. number of nodes J of the estimated solutions (a) and their gradients (b).

J	$\epsilon(u)$ **for Test 1**	$\epsilon(u)$ **for Test 2**	$\epsilon(\nabla u)$ **for Test 1**	$\epsilon(\nabla u)$ **for Test 2**
41	2.433E-10	5.421E-2	6.302E-05	1.976E-1
145	1.655E-10	2.241E-2	7.330E-06	1.091E-1
545	1.357E-10	1.789E-2	7.304E-06	8.084E-2
2113	6.308E-11	1.671E-2	6.819E-06	1.002E-1

Table 3. Relative errors on parallelogram vs. number of nodes J of the estimated solutions (a) and their gradients (b).

J	$\epsilon(u)$ **for Test 1**	$\epsilon(u)$ **for Test 2**	$\epsilon(\nabla u)$ **for Test 1**	$\epsilon(\nabla u)$ **for Test 2**
25	6.953E-10	0.0395	4.892E-09	0.2769
81	4.525E-10	0.1469	4.242E-09	0.5887
289	1.908E-10	0.2575	5.801E-09	1.3156
1089	5.961E-11	0.24514	8.989E-09	1.5559

The power function that corresponds to the error's dependency on the total count of nodes J is found to be $\epsilon \sim J^{-\tau/2} \sim h^{\tau}$. In other words, J has corresponded with a power function. Here h is an average linear size of the elements. Through our numerical calculations for Test 1, it yields $\tau \sim 1$ for both square and parallelogram domains.

This rate of convergence is similar to that of the non-localised Neumann-related BDIE in (Mikhailov & Mohamed, 2012).

We have relatively slow convergence in circular domain, i.e., $\tau \sim 0.7$. For Test 2, it is prevailed that $\tau \sim 0.9$, $\tau \sim 0.8$ and $\tau \sim -0.9$ for square domain, circle and parallelogram, respectively. Likewise, we have $\epsilon(\nabla u) \sim J^{-\tilde{\tau}/2} \sim h^{\tilde{\tau}}$ for gradient error. The results for Test 1 are $\tilde{\tau} \sim 0.05$, $\tilde{\tau} \sim -1.4$ and $\tilde{\tau} = -0.4$ for square, circle and parallelogram, respectively. While for Test 2, we obtained $\tilde{\tau} = 0.4$, $\tilde{\tau} = 0.3$ and $\tilde{\tau} = -0.9$ for square, circle and parallelogram, respectively.

The precision in Test 1 is far greater than in Test 2 due to the piece-wise bi-linear interpolation is exact on the linear solution. Therefore, only the error for integral operator approximation associated with numerical integration accuracy is concerned. On the contrary, the quadratic exact solution in Test 2 on the piece-wise bi-linear interpolation contributes to the total error.

IV. CONCLUSION

In this paper, we have presented the numerical results for the solution of Neumann LBDIEs with perturbed operator. The LBDIEs comprise boundary and domain integrals that deal with singular kernels. The singular kernels for boundary integrals are of logarithmic singularity and r^{-2} singularity. Whereas the singular kernels for domain integrals are of logarithmic and r^{-2} singularity. The LBDIEs contain the integrals with kernels that are only nonzero when $x \in \omega(y)$. This leads the LBDIEs to the sparse system of matrix equations. However, the matrix obtained from the perturbed integral operator is dense. The perturbation operator is added in order to make the Neumann LBDIEs has a unique solution. Therefore, the final matrix system of equations is not sparse that enables the use of iterative methods suitable for solving sparse matrix system.

The test problems that we have chosen are linear solution and quadratic solution. From the results we have obtained, it can be concluded that all methods in handling all singular boundary and domain integrals are reasonably good. The newly invented methods which are semi-analytic method in handling logarithmic singularity for boundary integration and semi-quadratic analytic method for r^{-2} singularity for boundary integration, are proven to give high accuracy. This can be seen for the test problem with a linear solution that only involved integration error. The test problem with quadratic solution involves integration and interpolation errors where interpolation error is a more prominent error.

For future research, it would be good if we could check how well both the semi-analytic method and semi-quadratic analytic method as compared to the use of the standard Gaussian quadrature i.e., Gauss-Legendre quadrature formula. Besides, it is excellent research if we can observe how these semi-analytic and semi-quadratic analytic methods act on various test problems, e.g., solutions involving complex variables.

In this paper, the singularity for double integration is handled by a transformation method i.e., a Duffy transformation. However, in this Duffy transformation, it is required to sub-mesh the related quadrilateral element into two triangular elements. Therefore, it is suggested that the procedure involved in the derivation of semi-analytic and semi-quadratic analytic methods for singular boundary integration are extended to the singular domain integration. It is proposed that this new method for singular domain integration if successfully derived can be a great alternative to the transformation method to handle singularity of double integration. The well-known transformation methods are e.g., singular integral are Duffy transformation and transformation.

Other future research that we may consider is to extend the use of the semi-analytic and semi-quadratic analytic methods to the BDIE/BDIDE obtained from the Helmholtz and heat conduction equations with variable coefficient.

It would be a good effort if we could extend our experiments to further test domains with various test problems to see how the results will react to the distortion of interior mesh elements. It is also suggested to attempt the use of triangular mesh elements instead of quadrilateral mesh elements.

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