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Comparison between Simultaneous and Sequential Movements in a Duopoly Game

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In general, a Stackelberg competition benefits the leader and the consumer due to making the first move in a sequential game. However, in this paper, sequential movement does not offer an advantage over simultaneous movement in a static duopoly under quantity competition with an isoelastic demand function. By introducing dynamics into the duopoly model through numerical simulations of simultaneous and sequential movements towards the Cournot equilibrium, there is evidence of first-mover advantage. However, simultaneous movement is advantageous over sequential movement in terms of better profits for both firms and lower prices for the consumer.

Keywords: Stackelberg; sequential; simultaneous; duopoly; isoelastic demand; Cournot

I. INTRODUCTION

A duopoly is a market dominated by two firms. Some examples of duopoly are Coca-Cola, Pepsi, Visa and Mastercard. Cournot was the first to introduce duopoly in 1838 when he formulated a mathematical model describing the quantity competition between two firms, which was then extended in 1883 by Bertrand, who proposed price competition instead. Other common characteristics of a duopoly are that firms produce homogeneous products, firms behave rationally so that they usually seek to maximise their profits. Decisions on quantities or prices are chosen simultaneously. In 1934, Stackelberg proposed the idea of firms moving sequentially instead of simultaneously. The firm that makes the first move is the leader, while the other firm is the follower (Varian, 2006).

Theoretical and experimental studies have shown that the Stackelberg model is better than the Cournot model by allowing higher profits for firms and lower prices for consumers (Anderson & Engers, 1992; Huck *et al.*, 2001). However, by introducing asymmetry into the duopoly (Colombo & Labrecciosa, 2019) or letting the number of firms be more than two (Zouhar & Zouharova, 2020), the Cournot model can be better than the Stackelberg. Usually, a second-mover advantage arises in a Stackelberg duopoly under price competition and product differentiation (Amir & Stepanova, 2006; Cheng & Tabuchi, 2010). However, under price competition and product differentiation, first-mover advantage can still happen by assuming that both firms prefer to be leaders (Kosuke *et al.*, 2017) or by considering a Corporate Social Responsibility (CSR) leadership model (Kopel, 2021).

In the studies mentioned above, the duopoly model is developed by assuming a generalisation of the linear demand (Anderson & Engers, 1992), a linear inverse demand (Huck *et al.*, 2001; Colombo & Labrecciosa, 2019; Zouhar & Zouharova, 2020; Cheng & Tabuchi, 2010; Kosuke *et al.*, 2017; Kopel, 2021), or the demand function is not specified (Amir & Stepanova, 2006). In this study, we consider an isoelastic inverse demand function, which is derived by assuming that the market is governed by the Cobb-Douglas utility function. This function is a representation of the Cobb-Douglas production that has been statistically tested using the 1947 United States Census data (Douglas, 1976).

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Under quantity competition, we compare the static Cournot duopoly to the static Stackelberg duopoly in terms of output level, profit, and price. Then, we consider repeated duopoly games by numerically simulating the simultaneous and sequential movements towards a Cournot equilibrium. The numerical results of the simultaneous and sequential movements are then compared to each other, and whether first or second-mover advantage arises is discussed.

II. STATIC DUOPOLY

We consider a market where two firms compete by each producing a single homogeneous product. The inverse demand function is:

$$p = \frac{1}{X}, \tag{1}$$

where p is the price, $X = x_1 + x_2$ is the total output of the market, and $x_i > 0$ is the level of output for firm $i = 1,2$. The identical production cost of each firm is $c_i x_i$, $c_i > 0$, so that the profit of each firm is:

$$\pi_i = \frac{x_i}{X} - c_i x_i. \tag{2}$$

Both firms compete by producing outputs that can maximise their profits. Therefore, by the first-order condition, solving the marginal profit:

$$\frac{\partial \pi_i}{\partial x_i} = \frac{x_i}{X^2} - c_i, \tag{3}$$

and making sure that the output is nonnegative, gives:

$$x_1 = r_1(x_2) = \begin{cases} \sqrt{\dfrac{x_2}{c_1}} - x_2, & 0 < x_2 < \dfrac{1}{c_1}, \\ 0, & x_2 > \dfrac{1}{c_1}, \end{cases}$$
$$x_2 = r_2(x_1) = \begin{cases} \sqrt{\dfrac{x_1}{c_2}} - x_1, & 0 < x_1 < \dfrac{1}{c_2}, \\ 0, & x_1 > \dfrac{1}{c_2}, \end{cases} \tag{4}$$

where r_i is called the reaction function of firm i to the other firm.

In a Cournot duopoly, firms move simultaneously. Solving (4) simultaneously gives the trivial solution $x_i = 0$, which we will ignore since $x_i > 0$, and:

$$x_1^* = \frac{c_2}{(c_1 + c_2)^2}, x_2^* = \frac{c_1}{(c_1 + c_2)^2}, \tag{5}$$

which are the level of outputs that will maximise the profits of firms 1 and 2.

In a Stackelberg duopoly game, firms move sequentially instead of simultaneously. Let firms 1 and 2 be the leader and follower, respectively. Firm 1 will move first by producing output at a level that will maximise its profit. So, it will consider r_2 in (4) by substituting it in π_1 to get:

$$\pi_1 = \sqrt{c_2 x_1} - c_1 x_1. \tag{6}$$

By the first order condition, solving the marginal profit:

$$\frac{\partial \pi_1}{\partial x_1} = \frac{1}{2}\sqrt{\frac{c_2}{x_1}} - c_1, \tag{7}$$

gives:

$$x_1^L = \frac{c_2}{4c_1^2}. \tag{8}$$

Then, firm 2 follows this move by substituting (8) in r_2 in (4) to obtain:

$$x_2^F = \frac{2c_1 - c_2}{4c_1^2}. \tag{9}$$

The output levels in (8) and (9) are the outputs that will maximise firm 1 and 2 profits in the Stackelberg case. By changing the roles so that firm 2 is the leader while firm 1 is the follower and following similar derivations, the outputs that will maximise the profits are:

$$x_1^F = \frac{2c_2 - c_1}{4c_2^2}, x_2^L = \frac{c_1}{4c_2^2}. \tag{10}$$

The profit-maximising outputs for the Cournot and Stackelberg duopolies are summarised in the first row of Table 1. The second row shows the corresponding total output of the market, and substituting these total outputs in (1) yields the corresponding market price in the third row. Lastly, the corresponding profits of each firm at different equilibriums are shown in the fourth row, where $k = c_1/c_2$.

Table 1. The outputs, prices, and profits at equilibriums.

Cournot Duopoly	**Stackelberg Duopoly (Firm 1 is leader)**	**Stackelberg Duopoly (Firm 2 is leader)**
$x_1^* = \dfrac{c_2}{(c_1+c_2)^2}$ $x_2^* = \dfrac{c_1}{(c_1+c_2)^2}$	$x_1^L = \dfrac{c_2}{4c_1^2}$ $x_2^F = \dfrac{2c_1 - c_2}{4c_1^2}$	$x_1^F = \dfrac{2c_2 - c_1}{4c_2^2}$ $x_2^L = \dfrac{c_1}{4c_2^2}$
$X^* = \dfrac{1}{c_1 + c_2}$	$X^{LF} = \dfrac{1}{2c_1}$	$X^{FL} = \dfrac{1}{2c_2}$
$p^* = c_1 + c_2$	$p^{LF} = 2c_1$	$p^{FL} = 2c_2$
$\pi_1^* = \dfrac{1}{(k+1)^2}$	$\pi_1^L = \dfrac{1}{4k}$ $\pi_2^F = \dfrac{(2k-1)^2}{4k^2}$	$\pi_1^F = \dfrac{(2-k)^2}{4}$

$\pi_2^* = \dfrac{k^2}{(k+1)^2}$	$\pi_2^L = \dfrac{k}{4}$

In a Cournot duopoly, the firm with the lower production cost should be able to produce more output than the other firm. In a Stackelberg duopoly, the leader is expected to benefit from making the first move by producing more output than the follower, thus generating more profit.

Proposition 1. If $c_1 < c_2$, then $x_1^* > x_2^*$, $x_1^L > x_2^F$, and $x_1^F > x_2^L$.

Proof. If $c_1 < c_2$ is true, then:

$$x_1^* - x_2^* = \frac{c_2 - c_1}{(c_1 + c_2)^2}$$

is positive, i.e., $x_1^* > x_2^*$. By comparing the outputs of the leader to the follower, we have:

$$x_1^L - x_2^F = \frac{c_2 - c_1}{2c_1^2} \text{ and } x_1^F - x_2^L = \frac{c_2 - c_1}{2c_2^2},$$

which are both positive for $c_1 < c_2$, which means that $x_1^L > x_2^F$, and $x_1^F > x_2^L$. ■

Proposition 2. In the case of:

i. Cournot duopoly, $\pi_1^* > \pi_2^*$ if $0 < k < 1$ or $c_1 < c_2$.
ii. Stackelberg duopoly where firm 1 is the leader, $\pi_1^L > \pi_2^F$ if $0.25 < k < 1$ or $0.25c_2 < c_1 < c_2$.
iii. Stackelberg duopoly where firm 2 is the leader, $\pi_2^L > \pi_1^F$ if $1 < k < 4$ or $c_2 < c_1 < 4c_2$.

Proof. If $\pi_1^* > \pi_2^*$, then:

$$\pi_1^* - \pi_2^* = 1 - k^2 = (1-k)(1+k) > 0,$$

and ignoring the negative root yields $0 < k < 1$.
If $\pi_1^L > \pi_2^F$, then:

$$\pi_1^L - \pi_2^F = -4k^2 + 5k - 1 = (1-4k)(k-1) > 0,$$

which leads to $0.25 < k < 1$.
If $\pi_2^L > \pi_1^F$, then:

$$\pi_2^L - \pi_1^F = -k^2 + 5k - 4 = (1-k)(k-4) > 0,$$

which leads to $1 < k < 4$.

Propositions 1 and 2 imply that in the case of simultaneous movement, the firm with the lower cost of production will produce a greater amount than the firm with the higher production cost. Similarly, in the sequential movement, the leader can only benefit from making the first move if its production cost is lower than the follower. Next, we compare the total outputs and prices between the Cournot and the Stackelberg duopolies.

Proposition 3. If $c_1 < c_2$, then $X^{LF} > X^*$, $X^* > X^{FL}$, $p^{LF} < p^*$, and $p^* < p^{FL}$.

Proof. If $c_1 < c_2$ is true, then:

$$X^{LF} - X^* = \frac{c_2 - c_1}{2c_1(c_1 + c_2)} \text{ and } X^* - X^{FL} = \frac{c_2 - c_1}{2c_2(c_1 + c_2)}$$

are both positive, so $X^{LF} > X^*$ and $X^* > X^{FL}$. Meanwhile,

$$p^{LF} - p^* = p^* - p^{FL} = c_1 - c_2$$

is negative since $c_1 < c_2$. So, $p^{LF} < p^*$ and $p^* < p^{FL}$. ■

Proposition 3 implies that in a market with sequential movement, the consumer has access to a larger quantity of output at a lower price if the leader has a lower production cost than the follower.

Taken together, Propositions 1, 2, and 3 show that lower production cost is more important than making the first move. In fact, if we simplify the problem by assuming that firms share the identical cost cx_i, we have:

$$x_i = \frac{1}{4c}, \pi_i = \frac{1}{4c}, i = 1,2, \qquad X = \frac{1}{2c}, p = 2c, \tag{11}$$

as the outputs, prices, and profits at the equilibrium for both Cournot and Stackelberg duopolies. This further confirms that neither firms nor consumers benefit from sequential movement. In the next section, we consider the situation of repeated or dynamic duopoly games to determine whether they might provide any advantage over static games in the case of sequential movement.

III. DYNAMIC DUOPOLY BY NUMERICAL SIMULATIONS

Differential games can be used to introduce dynamics into Cournot and Stackelberg's duopoly models (Colombo & Labrecciosa, 2019; Kańska & Wiszniewska-Matyszkiel, 2021). It involves solving a maximisation problem subject to sticky prices by defining a current-value Hamiltonian in the form of $H_i(t) = \pi_i(t) + \lambda_i(t)\dot{p}(t)$, where $\lambda_i(t)$ is the costate variable, and $\dot{p}(t)$ is the sticky price. In our case, the current-value Hamiltonian does not fulfil the sufficient condition for a concave Hamiltonian evaluated at the solution. Therefore, we attempt to introduce dynamics into the duopoly through numerical simulations of the simultaneous and sequential movements.

As shown in the previous section, production cost has a greater influence than type of movements on the output, price, and profit of the duopoly games. Since our main purpose is to observe the effect of different movements on the duopoly, we simplify the duopoly model by assuming firms share the identical cost $cx_i, c > 0$, so that (4) becomes

$$\begin{aligned} x_1 = r_1(x_2) &= \begin{cases} \sqrt{\dfrac{x_2}{c}} - x_2, & 0 < x_2 < \dfrac{1}{c}, \\ 0, & x_2 > \dfrac{1}{c}, \end{cases} \\ x_2 = r_2(x_1) &= \begin{cases} \sqrt{\dfrac{x_1}{c}} - x_1, & 0 < x_1 < \dfrac{1}{c}, \\ 0, & x_1 > \dfrac{1}{c}, \end{cases} \end{aligned} \tag{12}$$

Figure 1 is a representation of the reaction curves in (12), and the intersection of these two curves is the equilibrium $(1/4c\,, 1/4c)$.

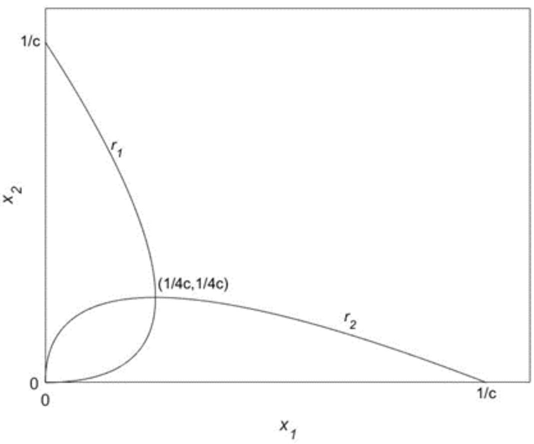
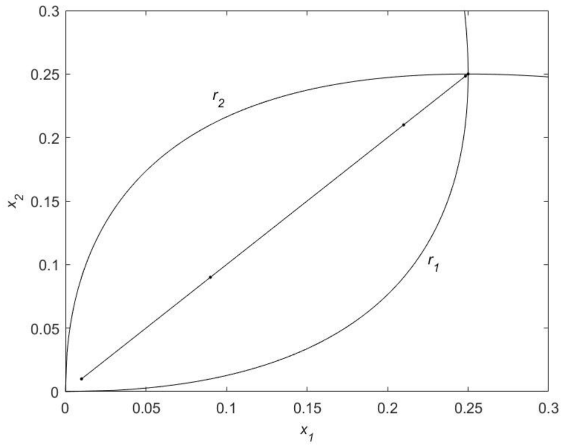
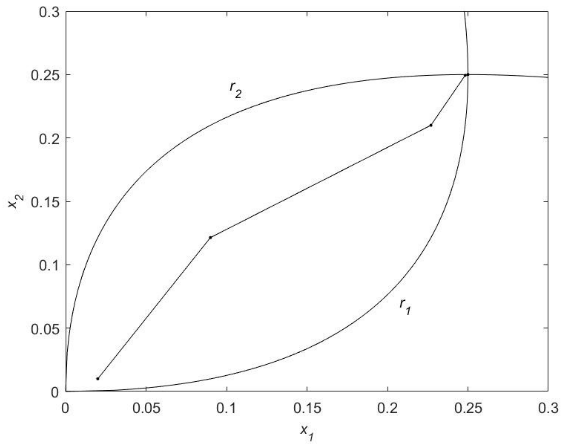


Figure 1. The reaction curves r_1 and r_2 in (12).

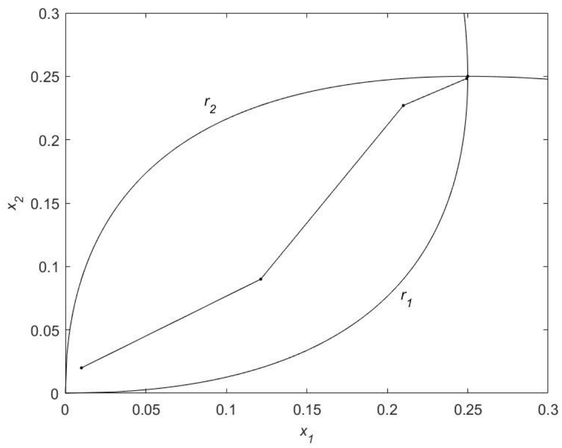
We assume that both firms are moving towards $(1/4c\,, 1/4c)$. For simplicity's sake, let $c = 1$. The numerical simulations of the simultaneous and sequential movements toward the equilibrium (0.25,0.25) are computed using (12) for three different initial outputs. These movements or adjustments are shown in Figures 2 and 3 for initial outputs (0.01,0.01), (0.02,0.01), and (0.01,0.02). For the sequential movements, firm 1 makes the first move.



(a) Initial output (0.01,0.01).

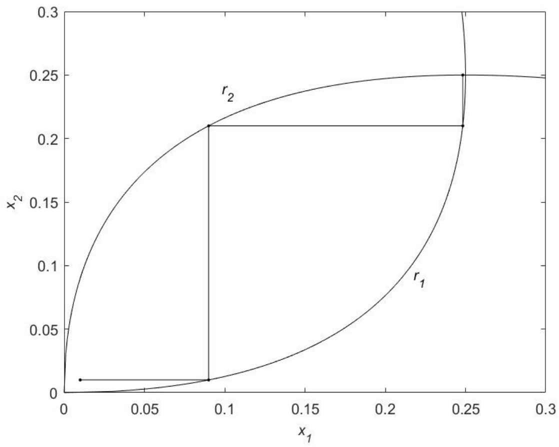


(b) Initial output (0.02,0.01).

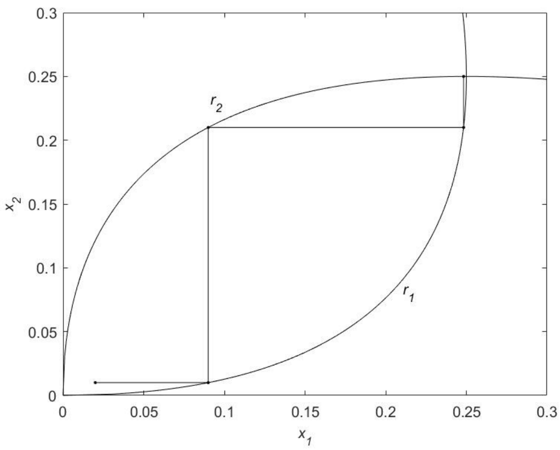


(c) Initial output (0.01,0.02).

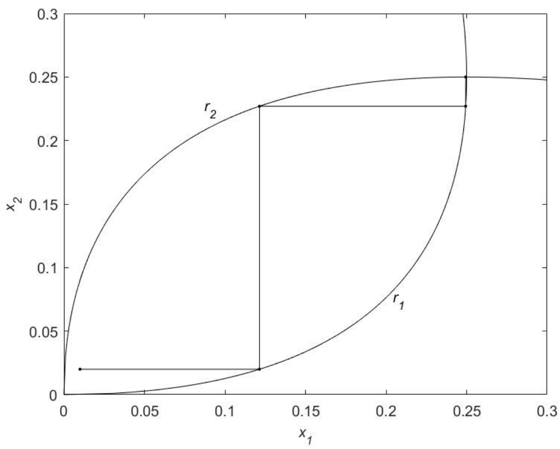
Figure 2. Simultaneous output adjustments for three different initial outputs.



(a) Initial output (0.01,0.01).



(b) Initial output (0.02,0.01).



(c) Initial output (0.01,0.02).

Figure 3. Sequential output adjustments for three different initial outputs.

Table 2 corresponds to the numerical simulations of the simultaneous movement in Figure 2 by showing the outputs of firms 1 and 2 and the corresponding total output, price, and profit at each period. A similar description applies to Table 3, corresponding to Figure 3. Note that for the simultaneous movement, both firms reached the equilibrium in period 5, and in the sequential movement, firm 2 arrives in period 5 while firm 1 arrives in period 6.

Table 2. Related values to simultaneous movement corresponding to Figure 2 for three different initial outputs.

t	$x_1(t)$	$x_2(t)$	$X(t)$	$p(t)$	$\pi_1(t)$	$\pi_2(t)$
	Initial Output $(0.01, 0.01)$					
1	0.01	0.01	0.02	50	0.49	0.49
2	0.09	0.09	0.18	5.5556	0.41	0.41
3	0.21	0.21	0.42	2.3809	0.29	0.29
4	0.2483	0.2483	0.4966	2.0136	0.2517	0.2517
5	0.25	0.25	0.5	2	0.25	0.25
	Initial Output $(0.02, 0.01)$					
1	0.02	0.01	0.03	33.3333	0.6467	0.3233
2	0.09	0.1214	0.2114	4.7304	0.3357	0.4529
3	0.2270	0.21	0.437	2.2883	0.2925	0.2705
4	0.2483	0.2494	0.4977	2.0092	0.2506	0.2517
5	0.25	0.25	0.5	2	0.25	0.25
	Initial Output $(0.01, 0.02)$					
1	0.01	0.02	0.03	33.3333	0.3233	0.6467
2	0.1214	0.09	0.2114	4.7304	0.4529	0.3357
3	0.21	0.227	0.437	2.2883	0.2705	0.2924
4	0.2494	0.2483	0.4977	2.0092	0.2517	0.2506
5	0.25	0.25	0.5	2	0.25	0.25

Table 3. Related values to sequential movement corresponding to Figure 3 for three different initial outputs.

t	$x_1(t)$	$x_2(t)$	$X(t)$	$p(t)$	$\pi_1(t)$	$\pi_2(t)$
	Initial Output $(0.01, 0.01)$					
1	0.01	0.01	0.02	50	0.49	0.49
2	0.09	0.01	0.1	10	0.81	0.09
3	0.09	0.21	0.3	30	0.21	0.49
4	0.2483	0.21	0.4583	2.1819	0.2935	0.2482
5	0.2483	0.25	0.4983	2.0068	0.25	0.2517
6	0.25	0.25	0.5	2	0.25	0.25
	Initial Output $(0.02, 0.01)$					
1	0.02	0.01	0.03	33.3333	0.6467	0.3233
2	0.09	0.01	0.1	10	0.8100	0.09
3	0.09	0.21	0.3	3.3333	0.2100	0.49
4	0.2483	0.21	0.4583	2.1819	0.2935	0.2482
5	0.2483	0.25	0.4983	2.0068	0.25	0.2517
6	0.25	0.25	0.5	2	0.25	0.25
	Initial Output $(0.01, 0.02)$					
1	0.01	0.02	0.03	33.3333	0.3233	0.6467
2	0.1214	0.02	0.1414	7.0721	0.7371	0.1214
3	0.1214	0.227	0.3484	2.8703	0.2271	0.4245
4	0.2494	0.227	0.4764	2.0991	0.2499	0.2495
5	0.2494	0.25	0.4994	2.0024	0.25	0.2506
6	0.25	0.25	0.5	2	0.25	0.25

Since the simulated data in Tables 2 and 3 are skewed, the median is used to measure the central value for the outputs, total output, price, and profits, as shown in Table 4. The sequential movement does offer an advantage to the first mover, as shown by Firm 1's greater profit than Firm 2 for all three initial values in the sequential movement column. However, both firms get better profits if they move simultaneously instead of sequentially. Also, simultaneous movement benefits the consumer in terms of access to a larger quantity of output at a lower price for all three initial outputs.

Table 4. Median values comparison between simultaneous and sequential movements from Tables 2 and 3.

Median	**Simultaneous**	**Sequential**
	Initial Output $(\mathbf{0.01, 0.01})$	
$\tilde{x}_1(t)$	0.21	0.1692
$\tilde{x}_2(t)$	0.21	0.21
$\tilde{X}(t)$	0.42	0.3792
$\tilde{p}(t)$	2.3809	6.09095
$\tilde{\pi}_1(t)$	0.29	0.27175
$\tilde{\pi}_2(t)$	0.29	0.25085
	Initial Output $(\mathbf{0.02, 0.01})$	
$\tilde{x}_1(t)$	0.227	0.16915
$\tilde{x}_2(t)$	0.21	0.21
$\tilde{X}(t)$	0.437	0.37915
$\tilde{p}(t)$	2.2883	2.7576
$\tilde{\pi}_1(t)$	0.2925	0.27175
$\tilde{\pi}_2(t)$	0.2705	0.25085
	Initial Output $(\mathbf{0.01, 0.02})$	
$\tilde{x}_1(t)$	0.21	0.1845
$\tilde{x}_2(t)$	0.227	0.227
$\tilde{X}(t)$	0.437	0.4124
$\tilde{p}(t)$	2.2883	2.4847
$\tilde{\pi}_1(t)$	0.2705	0.25
$\tilde{\pi}_2(t)$	0.2924	0.2503

IV. CONCLUSION

We have demonstrated that in a static game, a Stackelberg duopoly under quantity competition with isoelastic demand function only benefits the leader and the consumer if the leader's production cost is lower than the follower. In the special case when production cost is identical, the outcome is the same for both Cournot and Stackelberg duopoly. This shows that sequential movement does not offer any advantage over simultaneous movement. However, through numerical simulations of the simultaneous and sequential movements toward the Cournot equilibrium, there is evidence of a first-mover advantage. Nevertheless, simultaneous adjustment is still advantageous over sequential movement in terms of better profits for firms and lower prices with larger total output for consumers.

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