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Pre-Schmidt Mode Parameters in Quantum Entanglement

M.T. Selvam[1], M.S. Nurisya[1,2*] and S.K. Said Husain[3,4]

[1]*Laboratory of Computational Sciences and Mathematical Physics, Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia*

[2]*Department of Physics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia*

[3]*Laboratory of Cryptography, Analysis and Structure, Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia*

[4]*Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia*

We wrote this paper to study in detail the mathematical methods used in deriving the parameters p'_1, p'_2, x'_1, x'_2, M, K and $e^{2\eta}$ in the research papers by Han *et al.* (1999), Makarov (2018a), Han *et al.* (1993), and Han *et al.* (1995) to find the Schmidt modes Λ_k in quantum entanglement. Here, we have analysed and developed a thorough calculation in exploring the rationales behind the existence of these parameters in two and three-coupled harmonic oscillators. Various mathematical approaches were applied in the study, ranging from polynomials and linear algebra to trigonometry and the Pythagorean theorem. We found the parameters K and $e^{2\eta}$ using the matrices' determinants and eigenvalues. With these rationales in deriving the parameters K and $e^{2\eta}$ in the research papers for the two-coupled harmonic oscillators, we have formulated similar parameters for three-coupled harmonic oscillators as the conclusion of our study.

Keywords: Hamiltonian; kinetic energy; potential energy; matrices; trigonometry

I. INTRODUCTION

The Hamiltonian of a two-coupled harmonic oscillators system from a few research papers (Han *et al.,* 1999; Makarov, 2018a; Han *et al.,* 1993; Han *et al.,* 1995) were as follows:

$$H_1^2 = \frac{1}{2}\left(\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2 + Ax_1^2 + Bx_2^2 + Cx_1x_2\right), \qquad (1)$$

where m_k denotes different masses, p_k is the momentum variable, x_k is the coordinate variable with $k = 1, 2$, and A, B and C are the coupling constants. H_n^2 denotes the Hamiltonian for a two-coupled harmonic oscillator system, where n represents the sequence of the Hamiltonian.

The solution of the stationary Schrodinger equation with the Hamiltonian in Equation (1) is known and stated in a previous study (Han *et al.,* 1999). However, problems arise due to the complexity of calculating and analysing the quantum entanglement for such a system. Usually, the quantum entanglement is analysed for the ground state of an oscillator. According to Ekert and Knight (1995), this is because the Schmidt modes Λ_k that were used in the analysis of quantum entanglement was not calculated in a general form.

In the previous research papers (Han *et al.,* 1999; Makarov, 2018a; Han *et al.,* 1993; Han *et al.,* 1995), there are parameters p'_1, p'_2, x'_1, x'_2 and M in the *Kinetic Energy* portion. There is also an appearance of $\tan 2\alpha$ following the diagonalisation of the *Potential Energy* portion in the Hamiltonian. Later, there are two more parameters K and $e^{2\eta}$, in finding the Schmidt modes Λ_k. The "how and why" of the existence of these pre-Schmidt mode parameters and

*Corresponding author's e-mail: risya@upm.edu.my

tan 2α were not comprehensively explained in the research papers. Thus, they exhibit difficulty in understanding them.

Our work can be considered a mild *scoping review*. It is an exploratory project that systematically probes the literature on identifying the key concepts, theories, sources of evidence and gaps in the research. We identified the knowledge gap and hence explored the rationale behind the arrival of those parameters. However, we have limited our study to only as far as the parameters K and $e^{2\eta}$. Subsequently, using the two-coupled harmonic oscillators system from the research papers, our study expanded to a three-coupled harmonic oscillators system.

The Hamiltonian for a three-coupled harmonic oscillator designed using Equation (1) as guidance:

$$H_1^3 = \frac{1}{2}\left(\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2 + \frac{1}{m_3}p_3^2 + Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_1x_2 + Ex_1x_3 + Fx_2x_3\right), \quad (2)$$

where m_k denotes different masses of the three-coupled harmonic oscillators together with their respective momentums p_k and their coordinate variables x_k, with $k = 1, 2, 3$ and their coupling constants A, B, C, D, E and F. H_n^3 denotes the Hamiltonian for the three-coupled harmonic oscillators system, which n represents the sequence of the Hamiltonian.

In Section II, we designed the Methodology into two subsections: detailed calculation methods to find the kinetic and potential energy. Here, we investigate the parameters involved in two-coupled and three-coupled harmonic oscillator systems. In Section III, we showed the verdicts of our study. Additionally, we discussed various representations of the parameters, how the coordinate rotation angle α can influence these parameters, compared our work with other researchers and generalised for N -coupled harmonic oscillators system. In Section IV, we have concluded our work on the parameters of the pre-Schmidt mode.

II. MATERIALS AND METHOD

The ultimate goal here is to solve the puzzle of how the parameters K and $e^{2\eta}$ in the two-coupled harmonic oscillator system derived in the research papers of Han *et al.* (1999), Makarov (2018a), Han *et al.* (1993), and Han *et al.* (1995) and finally to extract the same parameters for the three-coupled harmonic oscillators system.

Before that, we attempted to close the gaps among equations where details were omitted in the original pieces of literature. The mini titles *Closing the Gap* and *Rationale for Parameters* explained those missing links appropriately in stages in the following sections. We successfully clarified the derivation of those parameters and gleaned the parameter K for the three-coupled harmonic oscillators system. However, the parameter $e^{2\eta}$ is futile for a three-coupled harmonic oscillators system due to its complexity in deriving and utilising it for further calculations.

Appropriate mathematical methods were applied to untie the knots in those research papers. Linear algebra is the primary tool used to uncouple the potential energy and the kinetic energy portions. The method of diagonalisation and determinants were used in explaining the parameters. Trigonometry and the Pythagorean theorem were also essential instruments in this study.

We have divided the methodology into the following two subsections, *Kinetic Energy Portion* and *Potential Energy Portion*. Each subsection elaborates on two-coupled and three-coupled harmonic oscillator systems accordingly.

A. Kinetic Energy

The left part of the Hamiltonian denotes the kinetic energy portion. In Equation (1), the kinetic energy is:

$$\frac{1}{2}\left(\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2\right),$$

and from Equation (2), the kinetic energy is:

$$\frac{1}{2}\left(\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2 + \frac{1}{m_3}p_3^2\right).$$

The following subsection explained the changes that took place in the kinetic energy portion of the two-coupled system in the works of literature. We utilised that to further develop a similar operation on the three-coupled system.

1. Two-coupled harmonic oscillators

We began the study on the two-coupled harmonic oscillators system. Han *et al.* (1995) jumps from Equation (1) to:

$$H_2^2 = \frac{1}{2M}(p_1'^2 + p_2'^2) + \frac{1}{2}(Ax_1^2 + Bx_2^2 + Cx_1x_2), \quad (3)$$

without clear details. Thus, we attempted to find the missing details next.

Closing the gap

For a start, one must diagonalise the potential energy portion, which is a second-degree polynomial (quadratic expression) in terms of x_1 and x_2 by referring to Han *et al.* (1999), Han *et al.* (1995), and Aravind (1989). This is possible by a single rotation using c*oordinate rotation*. However, the momentum variables will also undergo the same rotation. Hence, the uncoupling process of the potential energy by rotation alone will cause the coupling of the two kinetic energy terms.

For simplification, the kinetic energy portion in Equation (1) is brought to rotationally invariant form (Han *et al.,* 1999; McDermott & Redmount, 2004) so that:

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \tag{4}$$

where $\gamma_i = \left(\frac{m_j}{m_i}\right)^{1/4}$ with $i, j = 1, 2$ such that $i \neq j$. The same conditions for i and j as in Equation (4) can be applied to Equations (5), (8) and (9).

Next, one can perform matrix multiplication to Equation (4). By comparing both sides of the matrix equation, p_i can be written in terms of p_i'. Finally, by squaring them, we arrived at:

$$p_i^2 = \left(\frac{m_i}{m_j}\right)^{1/2} p_i'^2. \tag{5}$$

In addition, we worked on the conversion of the kinetic energy portion in Equation (1) by substituting Equation (5) as given below:

$$\frac{1}{2}\left(\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2\right) = \frac{1}{2}\left(\frac{1}{m_1}\left[\left(\frac{m_1}{m_2}\right)^{1/2} p_1'^2\right] + \frac{1}{m_2}\left[\left(\frac{m_2}{m_1}\right)^{1/2} p_2'^2\right]\right);$$

and then simplified to:

$$\frac{1}{2}\left(\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2\right) = \frac{1}{2}\left(\frac{1}{(m_1 m_2)^{1/2}}p_1'^2 + \frac{1}{(m_1 m_2)^{1/2}}p_2'^2\right).$$

When $M = (m_1 m_2)^{1/2}$, the above equation will transform into

$$\frac{1}{2}\left(\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2\right) = \frac{1}{2M}(p_1'^2 + p_2'^2). \tag{6}$$

We would like to emphasise here that in the literature by Han *et al.* (1999), there was a leap from Equation (4) direct to Equation (6). Hence, the above explanation was meant to fill the gap. Besides that, the reason for the appearance of the parameters in the main diagonal in Equation (4) is also unknown in the literature. Thus, the rationale for these parameters can be found next.

The rationale for Parameters $\boldsymbol{\gamma_i}$

The basic idea here is to convert the kinetic energy portion to a simpler form. To meet the purpose, the left-hand side of Equation (6) is to be written as its right-hand side. Therefore, for simplicity, the parameters m_1 and m_2 were predetermined as a new parameter M.

Firstly, the Equation (6) is transformed into matrix form, as the method given by Merdaci and Jellal (2020):

$$\frac{1}{2}\begin{pmatrix} p_1 & p_2 \end{pmatrix}\begin{pmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p_1' & p_2' \end{pmatrix}\begin{pmatrix} \frac{1}{M} & 0 \\ 0 & \frac{1}{M} \end{pmatrix}\begin{pmatrix} p_1' \\ p_2' \end{pmatrix}. \tag{7}$$

Next, we equated the determinants for both square matrices in Equation (7) and found $M = (m_1 m_2)^{1/2}$, which is then substituted into Equation (6). Subsequently, equating the terms p_i^2 with $p_i'^2$, square rooting them and changing the subject, we arrived at:

$$p_i' = \left(\frac{m_j}{m_i}\right)^{1/4} p_i. \tag{8}$$

Finally, p_1' and p_2' from Equation (8) can be written into matrix form exactly as Equation (4). So, we have elucidated above the relevance behind introducing the parameters γ_i in Equation (4).

In addition, the scale transformation above does vary the x_1 and x_2 variables too as explained by Han *et al.* (1999). When one attempts canonical transformations, the transformation will lead to:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{9}$$

where $\gamma_i = \left(\frac{m_i}{m_j}\right)^{1/4}$.

However, the scale transformations on the position variables are inversely proportional to those of their

conjugate momentum variables. This is due to Hamiltonian formalism where the position and momentum variables are independent variables. Thus, the canonical transformation will make arise to a unitary transformation in quantum mechanics.

Therefore, the Hamiltonian can be written as the Hamiltonian in Equation (3). In the next section, we explore into three-coupled harmonic oscillators system.

2. Three-coupled harmonic oscillators

For the three-coupled harmonic oscillators system, Equation (2) can be written as:

$$H_2^3 = \frac{1}{2M}(p_1'^2 + p_2'^2 + p_3'^2) + \frac{1}{2}(Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_1x_2 + Ex_1x_3 + Fx_2x_3). \quad (10)$$

Here, we can see that there is a sudden leap from Equation (2) to Equation (10). In the following subsection, we elaborate clearly on how we convert Equation (2) to Equation (10).

Closing the gap

Based on Equation (4), we brought the kinetic energy portion of Equation (2) to rotationally invariant form by doing the following:

$$\begin{pmatrix} p_1' \\ p_2' \\ p_3' \end{pmatrix} = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \quad (11)$$

where $\gamma_i = \left(\frac{m_j m_k}{m_i^2}\right)^{1/6}$ with $i, j, k = 1, 2, 3$ such that $i \neq j \neq k$. The same conditions for i, j and k as in Equation (11) can be applied for Equations (12), (15) and (16).

Next, we performed matrix multiplication to Equation (11). Then, we compared both sides of the matrix equation and wrote p_i in terms of p_i'. Finally, by squaring them we achieved the following result:

$$p_i^2 = \left(\frac{m_i^2}{m_j m_k}\right)^{1/3} p_i'^2. \quad (12)$$

In the next step, we worked on the conversion of the kinetic energy portion in Equation (2) by substituting Equation (12):

$$\frac{1}{2}\left(\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2 + \frac{1}{m_3}p_3^2\right) = \frac{1}{2}\left(\frac{1}{m_1}\left[\left(\frac{m_1^2}{m_2 m_3}\right)^{1/3} p_1'^2\right] + \frac{1}{m_2}\left[\left(\frac{m_2^2}{m_1 m_3}\right)^{1/3} p_2'^2\right] + \frac{1}{m_3}\left[\left(\frac{m_3^2}{m_1 m_2}\right)^{1/3} p_3'^2\right]\right);$$

and then simplified to:

$$\frac{1}{2}\left(\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2 + \frac{1}{m_3}p_3^2\right) = \frac{1}{2}\left(\frac{1}{(m_1 m_2 m_3)^{1/3}}p_1'^2 + \frac{1}{(m_1 m_2 m_3)^{1/3}}p_2'^2 + \frac{1}{(m_1 m_2 m_3)^{1/3}}p_3'^2\right).$$

Next, we substituted $M = (m_1 m_2 m_3)^{1/3}$, which also can be found by Merdaci and Jellal (2020), the above equation will transform into:

$$\frac{1}{2}\left(\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2 + \frac{1}{m_3}p_3^2\right) = \frac{1}{2M}(p_1'^2 + p_2'^2 + p_3'^2). \quad (13)$$

We will revisit Equation (13) later in the next part. We will look into the reason for the parameters in the main diagonal of Equation (11).

The rationale for Parameters $\boldsymbol{\gamma_i}$

Similar to the two-coupled harmonic oscillators system, the fundamental idea is to convert the kinetic energy portion to a simpler form as in Equation (13). Again, the parameters m_1, m_2 and m_3 were predetermined as a new parameter M.

We started by transforming Equation (13) into matrix form, similar to the method by Merdaci and Jellal (2020):

$$\frac{1}{2}\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p_1' & p_2' & p_3' \end{pmatrix} \begin{pmatrix} \frac{1}{M} & 0 & 0 \\ 0 & \frac{1}{M} & 0 \\ 0 & 0 & \frac{1}{M} \end{pmatrix} \begin{pmatrix} p_1' \\ p_2' \\ p_3' \end{pmatrix}. \quad (14)$$

In the next step, we equated the determinant for both square matrices in Equation (14) and derived $M = (m_1 m_2 m_3)^{1/3}$, which is then substituted into Equation (13). Thereafter, equating the terms p_i^2 with $p_i'^2$ square rooting them and changing the subjects and finally finding:

$$p_i' = \left(\frac{m_j m_k}{m_i^2}\right)^{1/6} p_i. \quad (15)$$

Eventually, p_1', p_2' and p_3' from Equation (15) can be rearranged into matrix form exactly as Equation (11). Hence, the discussion above explained the reason for introducing the parameters γ_i in Equation (11).

Based on Equation (9) and using the two-coupled harmonic system (Han *et al.,* 1999) as guidance, the scale transformation above does vary the x_1, x_2 and x_3 variables too. The attempt at canonical transformations will provide:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad (16)$$

where $\gamma_i = \left(\frac{m_i^2}{m_j m_k}\right)^{1/6}$.

Just as in the two-coupled harmonic oscillators system, the scale transformations on the position variables are inversely proportional to those of their conjugate momentum variables. Since the position and momentum variables are independent variables, the canonical transformation will lead to a unitary transformation. Thus, the Hamiltonian can be written as the Equation (10).

We have discussed the kinetic energy portion in detail in this subsection. In the next subsection, we explored the potential energy portion.

B. Potential Energy

The potential energy portion is on the right part of the Hamiltonian. In Equation (1), the kinetic energy is:

$$\frac{1}{2}(Ax_1^2 + Bx_2^2 + Cx_1x_2),$$

and from Equation (2), the kinetic energy reads:

$$\frac{1}{2}(Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_1x_2 + Ex_1x_3 + Fx_2x_3).$$

The following subsection elaborated on the transformations in the potential energy portion of a two-coupled system in the kinds of literature. Using that information as a tool, we worked on the potential energy for the three-coupled system.

1. *Two-coupled harmonic oscillators*

Using Equation (3), one can start to decouple Hamiltonian by performing the coordinate rotation (Aravind, 1989; Kim & Noz, 1991) to the potential energy portion (Han *et al.,* 1999; Han *et al.,* 1995; Han *et al.,* 1993) that is:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (17)$$

Here, the system is diagonalised when the angle of rotation α becomes:

$$\tan 2\alpha = \frac{C}{A-B}. \quad (18)$$

The potential energy portion from Equation (3) transformed into Equation (17) (Park, 2018; Han *et al.,* 1999; Han *et al.,* 1995; Han *et al.,* 1993). Finally, Equation (18) was presented in the paper. However, adequate steps in deriving the equations were not presented. The details can be found in the next part.

Closing the Gap

Before decoupling, the quadratic expression $Ax_1^2 + Bx_2^2 + Cx_1x_2$ from Equation (3) can be transformed into a (2×2) matrix as follows:

$$V^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} A & \frac{1}{2}C \\ \frac{1}{2}C & B \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad (19)$$

where V^2 is the potential matrix of a two-coupled system. After performing coordinate rotation to the square matrix, V^2 will become V_{new}^2 (Randles *et al.,* 2019):

$$V_{\text{new}}^2 = (R^T)(V^2)(R), \quad (20)$$

where $R = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ is the rotation coordinate and $R^T = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ is the transpose of R. When the matrices of R^T, V^2 and R were replaced in Equation (20), we arrived at:

$$V_{\text{new}}^2 = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} A & \frac{1}{2}C \\ \frac{1}{2}C & B \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Through this coordinate rotation, the kinetic energy portion in (3) remains invariant. So, one can perform decoupling by diagonalisation. After performing expansion and simplification, the resultant matrix V_{new}^2 is:

$$V_{\text{new}}^2 = \begin{bmatrix} \delta & \xi \\ \xi & \zeta \end{bmatrix}, \quad (21)$$

where

$$\delta = A\cos^2\alpha + B\sin^2\alpha + C\sin\alpha\cos\alpha\,,$$
$$\xi = \frac{C}{2}(\cos^2\alpha - \sin^2\alpha) + (B - A)\sin\alpha\cos\alpha\,,$$
$$\zeta = A\sin^2\alpha + B\cos^2\alpha - C\sin\alpha\cos\alpha\,.$$

Now, by setting the off-diagonal (ξ) in Equation (21) to zero, the equation obtained is:

$$\frac{C}{2}(\cos^2\alpha - \sin^2\alpha) + (B - A)\sin\alpha\cos\alpha = 0.$$

By applying the trigonometric functions $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ and $\sin 2\alpha = 2\sin\alpha\cos\alpha$, the equation is reduced to

$$\frac{C}{2}(\cos 2\alpha) + \left(\frac{B - A}{2}\right)(\sin 2\alpha) = 0.$$

Continuing further, the equation is simplified to Equation (18). A similar equation was also used by Barnett and Phoenix (1992) in the process of working with the Schmidt decomposition.

In Section III, we continued our discussion on different values of α and their effects on $\tan\alpha$. Different values of α also influence the parameters K and $e^{2\eta}$.

When we continued studying the papers from Han *et al.* (1993), Han *et al.* (1995), Makarov (2018a), and Han *et al.* (1999), we found that the following parameters were introduced:

$$K = \sqrt{AB - \frac{C^2}{4}},\ e^{2\eta} = \frac{A + B + \sqrt{(A-B)^2 + C^2}}{\sqrt{4AB - C^2}}. \quad (22)$$

Jellal *et al.* (2005) stated that the parameter $e^{2\eta}$ was analysed further and written in a slightly different form as:

$$e^{2\eta} = \frac{A + B + \sqrt{(A - B)^2 + C^2}}{2K}.$$

However, there were inadequate explanations given for the existence of these parameters in the pieces of literature mentioned above. Hence, we attempted to clarify this next.

The rationale for Parameters $\boldsymbol{K}$ *and* $\boldsymbol{e^{2\eta}}$

For a start, by utilising Equation (18), the image in Figure (1) was delineated.

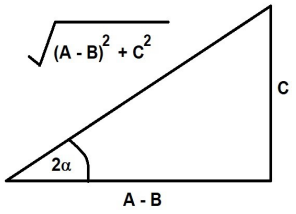

Figure 1. Relationship among the sides of a right-angled triangle with an angle of 2α.

It is discovered that the determinant and eigenvalues for the potential energy matrix in Equation (19) explained the two new parameters K and $e^{2\eta}$ in Equation (22). The length of the hypotenuse from Figure 1 can be seen as part of the numerator in the parameter $e^{2\eta}$. Despite the resemblance of the hypotenuse in parameter $e^{2\eta}$, it does not fully explain the parameters.

It was found that the parameter K is the square root of the determinant for the matrix V^2:

$$K = \sqrt{|V^2|} = \sqrt{AB - \tfrac{1}{4}C^2}\,, \quad (23)$$

where

$$\text{Det } V^2 = \begin{vmatrix} A & \frac{1}{2}C \\ \frac{1}{2}C & B \end{vmatrix} = AB - \left(\frac{1}{2}C\right)^2.$$

There's an important point that needs to be observed here, mathematically the Det $[V^2]$ is equal to the Det $[V_{\text{new}}^2]$. The parameter $e^{2\eta}$ on the other hand is the ratio of the eigenvalues λ of the matrix V^2 to the parameter K:

$$e^{2\eta} = \frac{\lambda}{K}. \quad (24)$$

The proof for the above argument is discussed next. Firstly, the determinant for $V^2 - I\lambda$ is to be found and then equate with zero, where I is the identity matrix. Thus, the characteristic polynomial for λ is obtained as below:

$$\lambda^2 - (A + B)\lambda + \left(AB - \frac{1}{4}C^2\right) = 0.$$

The characteristic polynomial for λ above is in the form of quadratic equation $a\lambda^2 + b\lambda + c = 0$. Hence, the λ can be found by using the formula of roots of the quadratic equation:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

By substituting $a = 1, \quad b = -(A + B)$ and $c = AB - \frac{1}{4}C^2$ followed by rearrangement, λ will be simplified to:

$$\lambda = \frac{A+B\pm\sqrt{(A-B)^2+C^2}}{2}. \tag{25}$$

Again, there's an important point here. The eigenvalues of V^2 and the eigenvalues of its rotational matrix, V^2_{new} are equal. However, the process of finding the eigenvalues using V^2_{new} is more tedious. Thus, for simplicity, V^2 was used in our study. Subsequently, the parameter $e^{2\eta}$ can be derived by:

$$e^{2\eta} = \frac{\lambda}{K} = \frac{\frac{A+B\pm\sqrt{(A-B)^2+C^2}}{2}}{\sqrt{AB - \frac{C^2}{4}}},$$

and finally simplified to:

$$e^{2\eta} = \frac{A+B\pm\sqrt{(A-B)^2+C^2}}{\sqrt{4AB-C^2}}. \tag{26}$$

The derivation above has proven the rationale for the parameter $e^{2\eta}$ in the literature mentioned. Nevertheless, the parameter $e^{2\eta}$ in Equation (26) has some differences compared to Equation (22) and in various kinds of literature too. Therefore, the parameter $e^{2\eta}$ will be discussed further in Section III.

Next, we extended the idea of the parameters K and $e^{2\eta}$ from two-coupled to three-coupled harmonic oscillators system in the following subsection.

2. Three-coupled harmonic oscillators

By referring to the two-coupled harmonic oscillators in the previous section, we start decoupling the Hamiltonian for Equation (10) by performing the coordinate rotation to the potential energy portion for the axes of x, y and z.

There are three angles of rotations φ, ϕ and θ which are associated with the x, y and z-axis respectively. These angles explain the rotations in the three-dimensional (3-D) spaces as given below:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{27}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{28}$$

and

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{29}$$

Before decoupling the Hamiltonian, the polynomial

$$Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_1x_2 + Ex_1x_3 + Fx_2x_3\,,$$

can be transformed into a (3×3) matrix as follows:

$$V^3 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} A & \frac{1}{2}D & \frac{1}{2}E \\ \frac{1}{2}D & B & \frac{1}{2}F \\ \frac{1}{2}E & \frac{1}{2}F & C \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{30}$$

where V^3 is the potential matrix of a three-coupled system. After performing coordinate rotation to the square matrix on the x, y and z-axes, V^3 will become V_t^3 as below:

$$V_t^3 = (R_t^T)(V^3)(R_t), \tag{31}$$

where R_t is the rotation coordinate on x, y and z–axes and R_t^T is the transpose of R_t with $t = x, y, z$. Therefore, it is easy to see that:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}, \quad R_x^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix};$$

$$R_y = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}, \quad R_y^T = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix};$$

$$R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_z^T = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Through this coordinate rotation, the kinetic energy portion in Equation (10) remains invariant. So, we performed decoupling by diagonalising. By expansion and simplification, the new matrices on the x, y and z-axis respectively are given as the following:

$$V_x^3 = \begin{bmatrix} A & \Upsilon & \varrho \\ \Upsilon & \mu & \Omega \\ \varrho & \Omega & \sigma \end{bmatrix}, \tag{32}$$

where

$$\mu = B\cos^2\varphi + C\sin^2\varphi + \frac{1}{2}F\sin 2\varphi,$$
$$\sigma = B\ \sin^2\varphi + C\ \cos^2\varphi - \frac{1}{2}F\ \sin\ 2\varphi,$$
$$\Upsilon = \frac{1}{2}(D\cos\varphi + E\ \sin\ \varphi),$$
$$\varrho = \frac{1}{2}(E\cos\varphi - D\ \sin\ \varphi),$$
$$\Omega = \frac{1}{2}[(C-B)\sin 2\varphi + F\cos 2\varphi];$$

$$V_y^3 = \begin{bmatrix} \pi & \iota & \vartheta \\ \iota & B & \nu \\ \vartheta & \nu & \rho \end{bmatrix}, \tag{33}$$

with

$$\pi = A\ \cos^2\phi + C\ \sin^2\phi + \frac{1}{2}E\ \sin\ 2\phi,$$
$$\rho = A\ \sin^2\phi + C\ \cos^2\phi - \frac{1}{2}E\ \sin\ 2\phi,$$
$$\iota = \frac{1}{2}(D\cos\phi + F\ \sin\ \phi),$$
$$\vartheta = \frac{1}{2}[(C-A)\sin 2\phi + E\ \cos\ 2\phi],$$
$$\nu = \frac{1}{2}(F\cos\phi - D\sin\phi);$$

and

$$V_z^3 = \begin{bmatrix} \varpi & \chi & \beta \\ \chi & \kappa & \tau \\ \beta & \tau & C \end{bmatrix}, \tag{34}$$

such that

$$\varpi = A\ \cos^2\theta + B\sin^2\theta + \frac{1}{2}D\ \sin\ 2\theta,$$
$$\kappa = A\sin^2\theta + B\cos^2\theta - \frac{1}{2}D\ \sin\ 2\theta,$$
$$\chi = \frac{1}{2}[(B-A)\sin 2\theta + D\ \cos\ 2\theta],$$
$$\beta = \frac{1}{2}(E\cos\theta + F\ \sin\ \theta),$$
$$\tau = \frac{1}{2}(F\cos\theta - E\ \sin\ \theta).$$

Now, by setting the off-diagonals (Ω, ϑ and χ) of the Equations (32), (33) and (34) to zero, we found:

$$\tan 2\varphi = \tfrac{F}{B-C}, \tag{35}$$
$$\tan 2\phi = \tfrac{E}{A-C}, \tag{36}$$
$$\tan 2\theta = \tfrac{D}{A-B}. \tag{37}$$

These angles φ, ϕ and θ will produce different effects on the three-coupled harmonic oscillators system depending on their values. Later, in Section III, we will continue our discussion on the different values of the angles and their effects on the three-coupled system.

Based on the rationale for the derivation of parameter K in Equation (22) as a guide, the new parameter K for the three coupled harmonic oscillators can be derived from the potential matrix in Equation (30):

$$K = \sqrt{|V^3|}$$
$$K = \sqrt{ABC + \tfrac{1}{4}(DEF - BE^2 - CD^2 - AF^2)}, \tag{38}$$

where

Det V^3

$$= A\begin{vmatrix} B & \frac{1}{2}F \\ \frac{1}{2}F & C \end{vmatrix} - \frac{1}{2}D\begin{vmatrix} \frac{1}{2}D & \frac{1}{2}F \\ \frac{1}{2}E & C \end{vmatrix} + \frac{1}{2}E\begin{vmatrix} \frac{1}{2}D & B \\ \frac{1}{2}E & \frac{1}{2}F \end{vmatrix}$$
$$= A\left(BC - \frac{1}{4}F^2\right) - \frac{1}{2}D\left(\frac{1}{2}CD - \frac{1}{4}EF\right) + \frac{1}{2}E\left(\frac{1}{4}DF - \frac{1}{2}BE\right);$$

the (2×2) matrices are the cofactors, $\left(BC - \frac{1}{4}F^2\right)$, $\left(\frac{1}{2}CD - \frac{1}{4}EF\right)$ and $\left(\frac{1}{4}DF - \frac{1}{2}BE\right)$ are the minors of A, $\frac{1}{2}D$ and $\frac{1}{2}E$ as in the first row of V^3, respectively.

With the rationale given previously for the derivation of parameters in Equation (22), the parameter $e^{2\eta}$ for the three-coupled harmonic oscillators system may seem possible to be designed by finding the ratio of eigenvalues of the matrix V^3 to the parameter K in Equation (38). However, as explained at the beginning of Section II, finding the eigenvalue λ for the three-coupled harmonic oscillators system is impossible due to its complexity. A further calculation involving parameter $e^{2\eta}$ for the three-coupled harmonic oscillators system cannot be pursued. Therefore, other means of possible calculations were designed for the three-coupled system by some researchers. This will be explained systematically in Section III.

In this section, we provided all the necessary reasons and proof regarding the calculations and parameters that lack details in the works of literature. Even though we have achieved our main aim, we wished to go the extra mile to investigate our work and compare it with the original reference literature in the next section.

III. RESULTS AND DISCUSSION

We did further studies on a few issues found in the literature. These issues are to be discussed next in a few subsections. In the beginning, we explored the possible values of α and their influences on two-coupled and three-coupled systems. Next, we analysed the various representations of the parameter $e^{2\eta}$

in the literature. Finally, we also showed the significance of the parameters K and $e^{2\eta}$ in the literature.

A. Possible Values of α and Their Influence on The Parameters K and $e^{2\eta}$

The values for the angle of rotation α may vary. The difference in the α value will affect the parameters K and $e^{2\eta}$ in two and three-coupled systems.

1. Two-coupled system

Let's revisit Equations (3) and (18) to discuss the two-coupled system. According to Han *et al.* (1999), the angle α may take various values. In Equation (18), if $\alpha = 0°$, then $\tan 2\alpha = 0$, therefore C will be zero. According to Equation (3), the $C = 0$ shows that the oscillators now become decoupled.

With the above information, we extended the concept with arbitrary values for α. When $\alpha = 90°$ and $\alpha = 180°$, $\tan 2\alpha = 0$, therefore $C = 0$ as well. When α takes these values, Equation (3) can be written as:

$$H_3^2 = \frac{1}{2M}(p_1'^2 + p_2'^2) + \frac{1}{2}(Ax_1^2 + Bx_2^2). \quad (39)$$

Thus, referring to Equation (22) the parameters become:

$$K = \sqrt{AB}\,, \qquad e^{2\eta} = \sqrt{\frac{A}{B}}. \quad (40)$$

However, referring to Equation (26), when $C = 0$ and due to the "−" sign preceding $\sqrt{(A-B)^2 + C^2}$, the parameter $e^{2\eta}$ will also take another value, which is $\sqrt{\frac{B}{A}}$.

There is an interesting Mathematical finding here. When the potential energy portion of Equation (39) is written in matrix form, the V^2 of Equation (19) will become $V_{(\alpha=0)}^2$ as defined below:

$$V_{(\alpha=0)}^2 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

The determinant of $V_{(\alpha=0)}^2$ is AB, which is the product of the main diagonal. Therefore, referring to Equation (23), the parameter K will be as stated in Equation (40). Another piece of information that can be found in the main diagonal is that the elements A and B are the eigenvalues. That being the case, the parameter $e^{2\eta} = \sqrt{\frac{A}{B}} = \sqrt{\frac{B}{A}}$ is stated by the formula in Equation (24).

As stated by Han *et al.* (1999), when $\alpha = 45°$, $\tan 2\alpha$ is undefined. Hence, the system is made up of two identical oscillators coupled together by the coupling constant C.

Referring to Equation (18), the angle $\alpha = 45°$ turns the denominator to zero, which means $A = B$. We arrived at the same result when we chose $\alpha = 135°$. Due to this situation, Equation (3) will transform into:

$$H_4^2 = \frac{1}{2M}(p_1'^2 + p_2'^2) + \frac{1}{2}(Ax_1^2 + Ax_2^2 + Cx_1x_2). \quad (41)$$

So, now the parameters in Equation (22) become:

$$K = \sqrt{A^2 - \frac{C^2}{4}}, \qquad e^{2\eta} = \sqrt{\frac{2A + C}{2A - C}},$$

Like in the earlier explanation, referring to Equation (26), when $A = B$ and due to the " − " sign in front of $\sqrt{(A-B)^2 + C^2}$, the parameter $e^{2\eta}$ can also take the value $\sqrt{\frac{2A-C}{2A+C}}$. Thus, Han *et al.* (1999) mentioned that η measures the strength of the coupling. Next, let's take a look at how similar situations are being dealt with in the three-coupled system.

2. Three-coupled system

We tried to venture into the three-coupled system with various values of angles φ, ϕ and θ. Since there are three different angles of rotation, we can analyse each angle with different characteristics. Hence, we should revisit Equations (10) and (35), (36), (37).

For a start, let's assume that all three angles are equal, where $\varphi = \phi = \theta$. Thus, we can take any equal values for them. If $\varphi = \phi = \theta = 0°, 90°$ or $180°$, then $\tan 2\varphi = \tan 2\phi = \tan 2\theta = 0$, therefore $D = E = F = 0$. This defined that the oscillators become decoupled. Following this, the Equation (10) can be rephrased as:

$$H_3^3 = \frac{1}{2M}(p_1'^2 + p_2'^2 + p_3'^2) + \frac{1}{2}(Ax_1^2 + Bx_2^2 + Cx_3^2). \quad (42)$$

By using Equation (38), we can calculate the expression for the parameter K, which is:

$$K = \sqrt{ABC}\,. \quad (43)$$

Parallel to the two-coupled system, the potential matrix for the Equation (42) can be written as $V^3_{(\varphi,\phi,\theta=0)}$ as follow:

$$V^3_{(\varphi,\phi,\theta=0)} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}.$$

The determinant of $V^3_{(\varphi,\phi,\theta=0)}$ is ABC, which is the product of the main diagonal. According to Equation (38), the parameter K will be as shown in Equation (43). An important fact that we could notice in $V^3_{(\varphi,\phi,\theta=0)}$ is that the elements A, B and C are the eigenvalues. In this situation, the parameter $e^{2\eta} = \sqrt{\frac{A}{BC}} = \sqrt{\frac{B}{AC}} = \sqrt{\frac{C}{AB}}$ by using the formula in Equation (24). On that account, the parameter $e^{2\eta}$ for the three-coupled system can only be found for $\varphi = \phi = \theta = 0°, 90°$ or 180°.

Next, let's assume that two of the angles are equal. For example, let's say that angles φ and ϕ are equal. We take either $0^o, 90^o$ or 180^o since the three angles will give the same value for $\tan 2\varphi$ and $\tan 2\phi$, which is zero. For the angle of θ, we take either 45° or 135° as their $\tan 2\theta$ will be undefined. Referring to the Equations (35) and (36), the values of $0^o, 90^o$ or 180^o for the angles, φ and ϕ will make the numerators F and E become zero.

Where else, referring to Equation (37) the values 45° or 135° for the angle θ will make the denominator, $A - B$ zero and hence, $A = B$. This defined that the first two oscillators are identical and coupled together by the D term. The third oscillator, however, become decoupled from the first and the second oscillators. Following this, the Equation (10) can be rephrased as:

$$H_4^3 = \frac{1}{2M}(p_1'^2 + p_2'^2 + p_3'^2) + \frac{1}{2}(Ax_1^2 + Ax_2^2 + Cx_3^2 + Dx_1x_2). \quad (44)$$

By referring to Equations (38) and (44), we can calculate the expression for the parameter K as follow:

$$K = \sqrt{A^2C - \frac{1}{4}CD^2}\,. \quad (45)$$

The opposite situation can be considered when φ and ϕ take either 45° or 135° where $\tan 2\varphi$ and $\tan 2\phi$ will be undefined. Now, by using the Equations (35) and (36), the denominators $B - C = A - C = 0$, thus $A = B = C$. Next, for angle θ we can assign the values $0^o, 90^o$ or 180^o which then results in $\tan 2\theta$ equal to zero. According to Equation (37), this causes D to be zero. This defined the system as consisting of three identical oscillators coupled together in two pairs The first and the third oscillators are coupled by the E term while the second and the third oscillators are coupled by the F term. The first two oscillators become decoupled. This situation is contrary to Equation (44). As a result, the Equation (10) will transform into:

$$H_5^3 = \frac{1}{2M}(p_1'^2 + p_2'^2 + p_3'^2) + \frac{1}{2}(Ax_1^2 + Ax_2^2 + Ax_3^2 + Ex_1x_3 + Fx_2x_3). \quad (46)$$

Referring to Equations (38) and (46), we can calculate the expression for the parameter K as follow,

$$K = \sqrt{A^3 - \frac{1}{4}A(E^2 + F^2)}\,. \quad (47)$$

A variety of similar situations can be designed by coupling the oscillators in different combinations which will produce different expressions for parameter K. This discussion will become very laborious if various values of angles are to be considered. Therefore, we limit our analysis to the angles and the effects on the parameters by considering the previous explanation as sufficient examples.

Finding $e^{2\eta}$ for the three-coupled harmonic oscillators is a challenging job, thus we discontinued the process in section II (B) (2). However, we found a paper that has similarity with our work on three-coupled systems where the derivation of parameters K and $e^{2\eta}$ shown by using the concept of limiting cases. This will be discussed next.

B. Similar Work Found on Parameters K and $e^{2\eta}$ in Literature

According to Merdaci and Jellal (2020), unlike a two-coupled system, it is not an easy task to analyse the entanglement in three or more particles due to the complicity of the problem. Thus, for a system of three-coupled harmonic oscillators, one has to consider three bipartitions (having a two-coupled system) and then look at the paring separately. This was done by using limiting cases.

The researchers fixed the physical parameters and choose coordinate variables by doing bipartitions of the three-coupled particles (x_1, x_2, x_3) to two-coupled particles such as (x_1, x_2), (x_1, x_3) and (x_2, x_3). The solution of a two-coupled system with variables (x_1, x_2) was found by limiting the

coupling constants E and F in Equation (2) to 0 ($E, F \to 0$). This restricted the Hamiltonian in Equation (2) to:

$$H_1^3 \to H_0^3 + \frac{1}{2}\left(\frac{1}{m_3}p_3^2 + Cx_3^2\right), \tag{48}$$

where H_0^3 is the Hamiltonian of the two-coupled (x_1, x_2)

$$H_0^3 = \frac{1}{2}\left(\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2 + Ax_1^2 + Bx_2^2 + Dx_1x_2\right). \tag{49}$$

By referring to Equations (35) and (36), taking $E, F \to 0$ will make the angles $\phi \to 0$ and $\varphi \to 0$. The potential matrix will be similar to the square matrix V^2 in Equation (19). Therefore, the parameters K and $e^{2\eta}$ found by Merdaci and Jellal (2020) are similar to Equation (22).

Similarly, the solution of a two-coupled system with variables (x_1, x_3) and (x_2, x_3) were found by limiting $D, F \to 0$ ($\theta \to 0$ and $\varphi \to 0$) and $D, E \to 0$ ($\theta \to 0$ and $\phi \to 0$), respectively in Equation (2). Hence, the parameters K and $e^{2\eta}$ were explained in Appendix B of Merdaci and Jellal (2020).

In the following part, we are going to take a quick look at the potential use of our results.

C. Potential Use of Our Results

Entanglement is a fundamental tool in solving many quantum mechanics problems such as quantum optics, quantum chemistry and also other fields of physics (de Souza Dutra, 2006). The entanglement makes it possible for the development in quantum information science to explain several quantum communications protocols such as quantum cryptography, quantum dense coding, quantum computing algorithms and quantum state teleportation (Makarov, 2018a; Merdaci & Jellal, 2020; Park, 2018; Kao & Chou, 2016).

Entanglement's strength is determined by the coupling forces between the particles (Merdaci & Jellal, 2020). Based on Equation (2), week coupling force is characterised by limiting the coupling constants $(D, E, F) \to 0$, where the angles $(\varphi, \phi, \theta) \to 0$ will cause the maximal value for the purity function, $P \to 1$. This shows that the system is completely separable, hence there are no entangled states since the entropy $S = 1 - P = 0$. Otherwise, strong coupling causes the minimal value for the purity function, $P \to 0$. Therefore, the system is maximally entangled due to $S = 1 - 0 = 1$.

Referring to Makarov (2018a) and Han *et al.* (1999), the parameters K and $e^{2\eta}$ are further to be utilised in wave function and also in the process of finding the Schmidt mode Λ_k. The Schmidt mode is one of the tools to measure the strength of entanglement. Thus, our results found in this study are important to the procedure of finding the wave function and the Schmidt mode.

Next, we will explore the parameter $e^{2\eta}$ presented in the works of literature.

D. Various Representations of The Parameter $e^{2\eta}$ in Literature

It is equally crucial to emphasise a few discoveries at this moment. Firstly, the parameter $e^{2\eta}$ was variously written in different papers. As for a beginning, the parameter was written as $e^{2\eta}$ (Makarov, 2018a). Next, $e^{-2\eta}$ were used (Han *et. al.,* 1995; Han *et al.,* 1993). We can also see the parameter were written differently, as e^{η} by Han *et al.* (1999). Anyhow, the symbols used may be different, but they all refer to the same thing. Hence, we have adapted $e^{2\eta}$ in our discussion.

Secondly, in the papers mentioned above, we also noticed that the "−" sign was omitted from the eigenvalues and only the "+" sign was used in the parameter $e^{2\eta}$ as stated in Equation (22). The literature might have chosen the "+" for simplicity purposes.

Thirdly, Makarov (2018a) stated that the parameter $e^{2\eta}$ was stated as:

$$e^{2\eta} = \frac{A+B+\frac{A-B}{|A-B|}\sqrt{(A-B)^2+C^2}}{\sqrt{4AB-C^2}}. \tag{50}$$

Here, the $\frac{A-B}{|A-B|}$ in Equation (50) seems to be the justification for the "$\pm$" sign in Equation (26). Different combinations of A and B produce different expressions for the parameter $e^{2\eta}$ as follows:

$$\text{If } A > B, \text{ then } e^{2\eta} = \frac{A+B+\sqrt{(A-B)^2+C^2}}{\sqrt{4AB-C^2}};$$

$$\text{If } A < B, \text{ then } e^{2\eta} = \frac{A+B-\sqrt{(A-B)^2+C^2}}{\sqrt{4AB-C^2}};$$

If $A = B$, then $e^{2\eta}$ is not defined, which relates to $\alpha = 45°$ as in our discussion in the earlier subsection where it resulted in Equation (41).

We have analysed two-coupled and three-coupled harmonic oscillators. To extend our work to the next level, we attempted to generalise our work for the N-coupled system. This will be discussed in the following section.

E. N-coupled Harmonic Oscillators System

We endeavoured in designing the Hamiltonian for N-coupled harmonic oscillators based on the Equations (1) and (2):

$$H_1^N = \frac{1}{2}\left[\sum_{k=1}^{N}\frac{1}{m_k}p_k^2 + \sum_{i=1}^{N}A_{ii}x_i^2 + \sum_{\substack{i,j=1\\(i<j)}}^{N}A_{ij}x_ix_j\right]; (A_{ij} = A_{ji}), \quad (51)$$

where m_k and p_k denote different masses and momentums respectively, coordinate variables x_i, x_j and their coupling constants A_{ii} and A_{ij} with i and j being rows and columns [Note: for A_{ii} both i's being rows and columns] of the potential square matrix. H_n^N denotes the Hamiltonian for the N-coupled harmonic oscillators system, where n represents the sequence of the Hamiltonian.

Later, the Hamiltonian will take the form similar to the two-coupled and the three-coupled systems in Equations (3) and (10):

$$H_2^N = \frac{1}{2M}\sum_{k=1}^{N}p'^2_k + \frac{1}{2}\left(\sum_{i=1}^{N}A_{ii}x_i^2 + \sum_{\substack{i,j=1\\(i<j)}}^{N}A_{ij}x_ix_j\right). \quad (52)$$

Hence the potential energy square matrix will take the form:

$$V^N = \begin{pmatrix} A_{11} & \frac{1}{2}A_{12} & \cdots & \frac{1}{2}A_{1(N-1)} & \frac{1}{2}A_{1N} \\ \frac{1}{2}A_{21} & A_{22} & & \frac{1}{2}A_{2(N-1)} & \frac{1}{2}A_{2N} \\ \vdots & & \ddots & \vdots & \\ \frac{1}{2}A_{(N-1)1} & \frac{1}{2}A_{(N-1)2} & \cdots & A_{(N-1)(N-1)} & \frac{1}{2}A_{(N-1)N} \\ \frac{1}{2}A_{N1} & \frac{1}{2}A_{N2} & & \frac{1}{2}A_{N(N-1)} & A_{NN} \end{pmatrix}.$$

Utilising this matrix potential, we attempted the derivation of the parameter K by using the formula found in the earlier section:

$$K = \sqrt{|V^N|}\,.$$

Here, the formula for K will vary depending on the determinant that relies on the value of N. A general rule to measure the determinant for V^N is similar to finding the determinant of V^3 in Equation (38). Let's take the first row of V^N to find the determinant as follows:

$$\begin{aligned} &|V^N| \\ &= (-1)^{(1+1)}A_{11}|minor\ of\ A_{11}| \\ &+ (-1)^{(1+2)}\frac{1}{2}A_{12}\left|minor\ of\ \frac{1}{2}A_{12}\right| + \cdots \\ &+ (-1)^{(1+(N-1))}\frac{1}{2}A_{1(N-1)}\left|minor\ of\ \frac{1}{2}A_{1(N-1)}\right| \\ &+ (-1)^{(1+N)}\frac{1}{2}A_{1N}\left|minor\ of\ \frac{1}{2}A_{1N}\right|; \end{aligned}$$

thus

$$\begin{aligned} &|V^N| \\ &= (-1)^{(1+1)}A_{11}\begin{vmatrix} A_{22} & \frac{1}{2}A_{23} & \cdots & \frac{1}{2}A_{2N} \\ & \vdots & \ddots & \vdots \\ \frac{1}{2}A_{N2} & \frac{1}{2}A_{N3} & \cdots & A_{NN} \end{vmatrix} \\ &+ (-1)^{(1+2)}\frac{1}{2}A_{12}\begin{vmatrix} \frac{1}{2}A_{21} & \frac{1}{2}A_{23} & \cdots & \frac{1}{2}A_{2N} \\ & \vdots & \ddots & \vdots \\ \frac{1}{2}A_{N1} & \frac{1}{2}A_{N3} & \cdots & A_{NN} \end{vmatrix} + \cdots \\ &+ (-1)^{(1+(N-1))}\frac{1}{2}A_{1(N-1)}\begin{vmatrix} \frac{1}{2}A_{21} & \cdots & \frac{1}{2}A_{2(N-2)} & \frac{1}{2}A_{2N} \\ \vdots & \ddots & & \vdots \\ \frac{1}{2}A_{N1} & \cdots & \frac{1}{2}A_{N(N-2)} & A_{NN} \end{vmatrix} \\ &+ (-1)^{(1+N)}\frac{1}{2}A_{1N}\begin{vmatrix} \frac{1}{2}A_{21} & A_{22} & \cdots & \frac{1}{2}A_{2(N-1)} \\ & \vdots & \ddots & \vdots \\ \frac{1}{2}A_{N1} & \frac{1}{2}A & \cdots & \frac{1}{2}A_{N(N-1)} \end{vmatrix}. \end{aligned}$$

This process will be repeated to reduce the dimension of the cofactors to (2×2) as the minors are easy to find. Thus, the number of attempts to reduce the cofactors depends on the value of N.

We have analysed in detail the parameters K and $e^{2\eta}$ involved in the literature. Questions may arise on the purpose and necessity of these parameters in the pre-Schmidt modes procedure. We will take a view on this in the next subsection.

F. The Significance of The Parameters K and $e^{2\eta}$ in The Literature

This section is just to review information in the pieces of literature and not for further discussion as the objective of this paper is only to derive the parameters.

We came across the parameters K and $e^{2\eta}$ in the literature in their process of deriving the Schmidt modes Λ_k on how the

parameters were utilised accordingly (Han *et al.,* 1999; 1995; 1993). The Hamiltonian in Equation (3) was written similarly as follows:

$$H_5^2 = \frac{1}{2M}(p_1'^2 + p_2'^2) + \frac{K}{2}(e^{2\eta}y_1^2 + e^{-2\eta}y_2^2), \quad (53)$$

with $y_1 = x_1\cos\alpha - x_2\sin\alpha$ and $y_2 = x_1\sin\alpha - x_2\cos\alpha$. It completed the diagonalisation process and the normal frequencies were given as follows:

$$\omega_1 = e^{\eta}\omega, \qquad \omega_2 = e^{-\eta}\omega,$$

with $\omega = \sqrt{\frac{K}{M}}$. Then, the author pursued his work from there.

On the other hand, the parameters were introduced but their usage was discontinued for the reasons quoted as "not a simple parametrisation" and "not the most convenient for analysing results" especially $e^{2\eta}$ (Makarov, 2018a). Instead, the researcher used the properties of trigonometric functions to convert the Equation (53) to:

$$H_6^2 = \frac{1}{2M}(p_1'^2 + p_2'^2) + \frac{1}{2}(A'y_1^2 + B'y_2^2); \quad (54)$$

$A' = A - \frac{C}{2}\tan\alpha$, $B' = B + \frac{C}{2}\tan\alpha$, and $\tan\alpha = \frac{\epsilon}{|\epsilon|}\sqrt{\epsilon^2+1} - \epsilon$, with $\epsilon = \frac{B-A}{C}\epsilon$. The paper then continues with its objective.

IV. CONCLUSION

Systems of two and three-coupled harmonic oscillators have been studied on a step-by-step basis. Our work was presented to fill up the gaps that are lacking in a few research papers. We have clarified the missing pieces for two main parameters K and $e^{2\eta}$ for the two-coupled harmonic oscillators system. By using the knowledge and guidance we gained from this two-coupled system, we developed our work into a three-coupled harmonic oscillators system. Though our study accomplished by finding those parameters, we did not stop there, instead went a little further and discussed some issues found during our process of writing this paper.

A Hamiltonian operator is a tool used in the research of quantum entanglement and various branches of sciences. It can be used for analysis and calculation without involving tedious numerical calculations, especially for large quantum numbers. The Schmidt decomposition is a convenient mathematical tool for measuring the nature of quantum entanglement (Ekert & Knight, 1995). One can use the value of the Schmidt modes Λ_k to calculate the measure of the quantum entanglement of the system (Makarov, 2018b). However, we realised that quantum entanglement using the Schmidt modes cannot be obtained for multi-component systems i.e., more than a two-coupled system. Other methods of measurement need to be found. An example proposed by Makarov (2018a) is "negativity" referring to Vidal and Werner (2002), and Galve *et al.* (2010).

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