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Dual Solutions in Nanofluids Passing a Moving Plate with Thermal Radiation and Stability Analysis

Nurul Shahirah Mohd Adnan[1*], Norihan Md Arifin[1,2], Norfifah Bachok[1,2] and Fadzilah Md Ali[1,2]

[1]*Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia*
[2]*Department of Mathematics, Faculty of Science, University Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia*

The present study deals with the flow and heat transfer in nanofluids over a moving plate in the present of thermal radiation. The governing boundary layer equations in the form of nonlinear partial differential equations are transformed into a system of nonlinear ordinary differential equations by using the similarity transformations method. Then, the obtained equations are solved numerically using the bvp4c function in MATLAB. Numerical results show that dual solutions exist for certain range of the controlling parameter. Stability analysis is employed to identify which solution is stable and valid physically. Results from the stability analysis shows that the first solution is stable and physically realizable while the second solution is unstable and not physically realizable.

Keywords: dual solutions, nanofluids, moving plate, stability analysis

I. INTRODUCTION

Nowadays, an innovative method in improving heat transfer has been proposed. This method known as nanofluid that was first introduced by (Choi, 1995). Nanofluid is a fluid containing nanometer-sized (less than 100nm) particles namely nanoparticles. Nanofluids also show good performance in transferring heat. In view of its applications, nanofluid has the biggest potential of being a new generation of coolants in automative applications. Besides, it is also helpful in cancer imaging and drug delivery for cancer therapeutics in biomedical industries (Das *et al.* 2006; Wong *et al.* 2010).

It is well known that thermal radiation plays an important role in controlling heat transfer process. Due to this reason, considerable research in the area of thermal radiation had been proposed recently by (Khan *et al.* 2012; Khan *et al.* 2014; Gaffar *et al.* 2017). The existence of dual or multiple solutions on the flow behavior have become a question which solution is stable and otherwise. Hence, (Merkin, 1985) have developed a method to solve this problem. After that, many researchers used the implemented method and they found that the first solution is always in stable state while the other is not (Sharma *et al.* 2014; Ishak, 2014; Junoh *et al.* 2018).

The purpose of this present study is to extend the work done by (Bachok *et al.* 2012) by analyze the effect of thermal radiation in nanofluids passing moving plate. Solving this problem gives dual solutions. Hence, we proceed the stability analysis to identify either the first solution or second solution is practically acceptable.

II. MATHEMATICAL FORMULATION

In the present paper, we consider the problem of laminar boundary layer flow that passing moving flat surface in a water-based nanofluid. We consider three different types of nanoparticles which are copper (Cu), titania (TiO_2) and alumina (Al_2O_3). We assumed the plate moves in the same or opposite direction to the free stream and both with constant velocities. The boundary layer equations given by:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

*Corresponding author's e-mail: iera_adnan@yahoo.com

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu_{nf}}{\rho_{nf}}\frac{\partial^2 u}{\partial y^2} \quad (2)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_{nf}\frac{\partial^2 T}{\partial y^2}-\frac{1}{\left(\rho C_p\right)_{nf}}\frac{\partial q_r}{\partial y} \quad (3)$$

along with the boundary conditions

$$\begin{aligned} &u=U_w,\ \ v=0,\ \ T=T_w,\ \text{at}\ \ y=0\\ &u\to U_\infty,\ \ T\to T_\infty,\ \text{as}\ \ y\to\infty \end{aligned} \quad (4)$$

Here, U_w and U_∞ are constant. U_w represents plate velocity while U_∞ corresponds the plate velocity. Next, u and v are the velocity components along the $x-$ and $y-$ directions, respectively. T is the nanofluid temperature. Further, μ_{nf}, α_{nf} and ρ_{nf} are viscosity, thermal diffusivity and density of nanofluid, respectively, as given by (Oztop& Abu-Nada, 2008) below:

$$\begin{aligned} &\alpha_{nf}=\frac{k_{nf}}{\left(\rho C_p\right)_{nf}},\ \ \mu_{nf}=\frac{\mu_f}{\left(1-\varphi\right)^{2.5}},\\ &\rho_{nf}=\left(1-\varphi\right)\rho_f+\varphi\rho_s,\\ &\frac{k_{nf}}{k_f}=\frac{\left(k_s+2k_f\right)-2\varphi\left(k_f-k_s\right)}{\left(k_s+2k_f\right)+\varphi\left(k_f-k_s\right)} \end{aligned} \quad (5)$$

Here, $(\rho C_p)_{nf}$ refer to the heat capacity of the nanofluid.k_{nf}, k_f and k_s represents the thermal conductivities of nanofluid, fluid and solid fractions, respectively. Further,$\rho_f$is the fluid density andρ_s is the solid fractions density.

The similarity solution for Equations (1)-(4) of the following form will be examined.

$$\begin{aligned} &\eta=\left(\frac{U}{\nu_f x}\right)^{1/2} y,\ \psi=\left(\nu_f x U\right)^{1/2} f(\eta),\\ &\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}, \end{aligned} \quad (6)$$

where$U = U_w + U_\infty$is the composite velocity. Then, ψ is the stream function that can be define as below:

$$u=\frac{\partial\psi}{\partial y},\ \ v=-\frac{\partial\psi}{\partial x} \quad (7)$$

Substituting Equation (6) into basic Equations (2) and (3) reduces this to the following nonlinear ordinary differential equations given by:

$$\frac{1}{\left(1-\varphi\right)^{2.5}\left(1-\varphi+\varphi\,\rho_s/\rho_f\right)}f'''+\frac{1}{2}ff''=0 \quad (8)$$

$$\frac{1}{\Pr}\frac{1}{1-\varphi+\varphi\left(\rho C_p\right)_s/\left(\rho C_p\right)_f}\left(\frac{k_{nf}}{k_f}+\frac{4}{3}R\right)\theta''+\frac{1}{2}f\theta'=0 \quad (9)$$

where primes indicate the differentiation with respect to η . $\Pr$ is the Prandtl number and R is the radiation parameter. All these parameters are defined as below:

$$\Pr=\frac{v_f}{\alpha_f},\ \ R=\frac{4\sigma^* T_\infty^3}{k_f k^*} \quad (10)$$

The boundary conditions in (4) become:

$$\begin{aligned} &f(0)=0,\ \ f'(0)=\lambda,\ \ \theta(0)=1,\\ &f'(\eta)\to 1-\lambda,\ \ \theta(\eta)\to 0\ \text{as}\ \eta\to\infty \end{aligned} \quad (11)$$

where $\lambda=U_w/U$ represents the velocity ratio parameter.

The physical quantities of interest are the skin friction coefficient C_f and the local Nusselt number Nu_x which are defined as below:

$$C_f=\frac{\tau_w}{\rho_f U^2},\ \ Nu_x=\frac{x q_w}{k_f\left(T_w-T_\infty\right)}, \quad (12)$$

where τ_w represents the surface shear stress while q_w represents the surface heat flux and are defined by

$$\tau_w=\mu_{nf}\left(\frac{\partial u}{\partial y}\right)_{y=0},\ \ q_w=-k_{nf}\left(\frac{\partial T}{\partial y}\right)_{y=0}, \quad (13)$$

Here, μ_{nf} is the dynamic viscosity of nanofluids and k_{nf} is the thermal conductivity of the nanofluids. By using the similarity variables in (6), we obtain:

$$C_f \mathrm{Re}_x^{1/2} = \frac{1}{(1-\varphi)^{2.5}} f''(0), \tag{14}$$

$$Nu_x / \mathrm{Re}_x^{1/2} = -\frac{k_{nf}}{k_f}\theta'(0), \tag{15}$$

where $\mathrm{Re}_x = Ux/\upsilon_f$ represents the local Reynolds number.

Table 1. Thermophysical properties of fluid and nanoparticles

Physical properties	Fluid phase (water)	Cu	Al_2O_3	TiO_2
$C_p(J/kgK)$	4179	385	765	686.2
$\rho(kg/m^3)$	997.1	8933	3970	4250
$k(W/mK)$	0.613	400	40	8.9538

III. STABILITY ANALYSIS

It has been discovered in (Weidman *et al.* 2006) that the second solution for the flow that passing a moving plate are unstable and not physically realizable. Meanwhile, the first solutions are stable and physically realizable. Then, to test these features, we first need to consider the unsteady case for Equations (2) and (3) by introducing the new dimensionless variable, $\tau = (U/x)t$.

From Equation (6), the new similarity solutions in terms of η and τ are as below:

$$\eta = \left(\frac{U}{\nu_f x}\right)^{1/2} y,\ \psi = (\nu_f x U)^{1/2} f(\eta,\tau), \quad \theta(\eta,\tau) = \frac{T-T_\infty}{T_w - T_\infty},\ \tau = \frac{U}{x}t. \tag{16}$$

Substituting Equation (16) into Equations (2) and (3) to become:

$$\frac{1}{(1-\varphi)^{2.5}\left(1-\varphi+\varphi\rho_s/\rho_f\right)}\frac{\partial^3 f}{\partial\eta^3} + \frac{1}{2} f\frac{\partial^2 f}{\partial\eta^2} - \frac{\partial^2 f}{\partial\eta\partial\tau} = 0 \tag{17}$$

$$\frac{1}{\Pr}\frac{1}{1-\varphi+\varphi(\rho C_p)_s/(\rho C_p)_f}\left(\frac{k_{nf}}{k_f}+\frac{4}{3}R\right)\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{2} f\frac{\partial\theta}{\partial\eta} - \frac{\partial\theta}{\partial\tau} = 0 \tag{18}$$

subject to the boundary conditions

$$f(0,\tau)=0,\ \frac{\partial f}{\partial\eta}(0,\tau)=\lambda,\ \theta(0,\tau)=1, \quad \frac{\partial f}{\partial\eta}(\eta,\tau)\to 1-\lambda,\ \theta(\eta,\tau)\to 0,\ \text{as}\ \eta\to\infty \tag{19}$$

To determine the stability of the solution $f = f_0(\eta)$ and $\theta = \theta_0(\eta)$ satisfying the boundary value problem (17)-(19), we write as below

$$f(\eta,\tau) = f_0(\eta) + e^{-\gamma\tau}F(\eta,\tau), \quad \theta(\eta,\tau) = \theta_0(\eta) + e^{-\gamma\tau}G(\eta,\tau) \tag{20}$$

where γ is an unknown eigenvalue while $F(\eta,\tau)$ and $G(\eta,\tau)$ are small relative to $f_0(\eta)$ and $\theta_0(\eta)$. Then, substituting Equation (20) into Equations (17)-(19) yields the following:

$$\frac{1}{(1-\varphi)^{2.5}\left(1-\varphi+\varphi\rho_s/\rho_f\right)}\frac{\partial^3 F}{\partial\eta^3} + \frac{1}{2} f_0\frac{\partial^2 F}{\partial\eta^2} + \frac{1}{2} f_0'' F + \gamma\frac{\partial F}{\partial\eta} - \frac{\partial^2 F}{\partial\eta\partial\tau} = 0 \tag{21}$$

$$\frac{1}{\Pr}\frac{1}{1-\varphi+\varphi(\rho C_p)_s/(\rho C_p)_f}\left(\frac{k_{nf}}{k_f}+\frac{4}{3}R\right)\frac{\partial^2 G}{\partial\eta^2} + \frac{1}{2} f_0\frac{\partial G}{\partial\eta} + \frac{1}{2}F\theta_0' + \gamma G - \frac{\partial G}{\partial\tau} = 0 \tag{22}$$

alongside the boundary conditions

$$F(0,\tau)=0,\ \frac{\partial F}{\partial\eta}(0,\tau)=0,\ G(0,\tau)=0, \quad \frac{\partial F}{\partial\eta}(\eta,\tau)\to 0,\ G(\eta,\tau)\to 0\ \text{as}\ \eta\to\infty \tag{23}$$

After that, we set $\tau = 0, F = F_0(\eta)$ and $G = G_0(\eta)$ to obtain the following:

$$\frac{1}{(1-\varphi)^{2.5}\left(1-\varphi+\varphi\rho_s/\rho_f\right)}F_0''' + \frac{1}{2}f_0F_0'' + \frac{1}{2}f_0''F_0 + \gamma F_0' = 0 \quad (24)$$

$$\frac{1}{\Pr}\frac{1}{1-\varphi+\varphi(\rho C_p)_s/(\rho C_p)_f}\left(\frac{k_{nf}}{k_f}+\frac{4}{3}R\right)G_0'' + \frac{1}{2}f_0G_0' + \frac{1}{2}F_0\theta_0' + \gamma G_0 = 0 \quad (25)$$

along with the conditions as follows:

$$F_0(0)=0,\ F_0'(0)=0,\ G_0(0)=0,\quad F_0'(\eta)\to 0,\ G_0(\eta)\to 0 \ \text{ as } \ \eta\to\infty \quad (26)$$

The stability of the problem can be tested via the smallest eigenvalue, γ. Therefore, the condition $F_0'(\eta)\to 0$ as $\eta\to\infty$ has been put at rest as suggested by (Haris *et al.* 2009).

IV. RESULTS AND DISCUSSIONS

To ensure the accuracy of the numerical method used in this paper, we made comparison results with those reported by (Bachok *et al.* 2012) when $R=0$ as stated in Table 2 below and it is worth mentioning that the numerical results which have beenobtained using bvp4c function in MATLAB software is a great agreement.

To test the accuracy of our code, we compute the numerical results for this problem. To solve these boundary value problems, the system of nonlinear equations is necessary to reduce to a system of first order ordinary differential equations. In this present paper, we consider for three different types of nanoparticles which are copper (Cu), titania (TiO_2) and alumina (Al_2O_3) with water as the base working fluid. Then, following (Oztop& Abu-Nada, 2008), the range of φ is between zero and 0.2 $(0\le\varphi\le 0.2)$ where $\varphi=0$ representing the regular fluid, and Prandtl number is equal to $6.2, \Pr = 6.2$ (water).

Table 2. Values of $f''(0)$ for some values of φ and λ for Cu. ([ ] refer to the second solution)

φ	λ	Bachok *et al.*, 2012	Present result
0	-0.5	0.3979 [0.1710]	0.39784 [0.17095]
	-0.4	0.4357 [0.0834]	0.43559 [0.08181]
	0	0.3321	0.33205
0.1	-0.5	0.4674 [0.2009]	0.46737 [0.20091]
	-0.2	0.4844 [0.0134]	0.48442 [0.01343]
	0	0.3901	0.39008
0.2	-0.5	0.4846 [0.2083]	0.48459 [0.20832]
	-0.2	0.5023 [0.0139]	0.50228 [0.01392]
	0	0.4045	0.40446

We get the numerical results as stated in Table 3 when we set values of $\varphi=0$ and $R=0.2$. From the table, we found that the values of $f''(0)$ and $-\theta'(0)$ for first and second solution for three types of nanoparticles are same even though we used various values of φ and λ. Hence, we just consider for Cu to generate the figures.

Table 3. Values of $f''(0)$ and $-\theta'(0)$ for some values of λ with different nanoparticle when $\varphi=0$ and $R=0.2$. ([ ] refer to the second solution)

Nanoparticle	λ	$f''(0)$	$-\theta'(0)$
Cu	-0.51	0.38911 [0.18480]	0.07982 [0.00118]
	-0.52	0.37828 [0.20073]	0.06632 [0.00185]

	-0.53	0.36421 [0.21993]	0.05227 [0.00305]
Al_2O_3	-0.51	0.38911 [0.18480]	0.07982 [0.00118]
	-0.52	0.37828 [0.20073]	0.06632 [0.00185]
	-0.53	0.36421 [0.21993]	0.05227 [0.00305]
TiO_2	-0.51	0.38911 [0.18480]	0.07982 [0.00118]
	-0.52	0.37828 [0.20073]	0.06632 [0.00185]
	-0.53	0.36421 [0.21993]	0.05227 [0.00305]

Figure 1 and Figure 2 show the variation of $f''(0)$ and $-\theta'(0)$ for Cu. From both Figures, it clearly shown that dual solutions exist in certain range value of $\lambda(\lambda_c > -0.5482)$ and there is no solution when $\lambda_c < -0.5482$. Further, Figures 3-5 are drawn to support the existing of dual solutions as shown in Table 3 and Figure 1 and 2 as well. These profiles are satisfied the boundary conditions (11) asymptotically. Therefore, it is also supporting the validity of the numerical results. Figure 3 shows that when radiation parameter, R increases, the thermal boundary layer thickness increases in first solution. Consequently, the heat transfer rate at the surface decreases in the present of radiation parameter. It is corresponding to the effect of the radiation is to decrease the rate of energy transport into the fluid, thus reducing the temperature of the fluid. Here, we noted that radiation parameter influenced the rate of heat transfer.

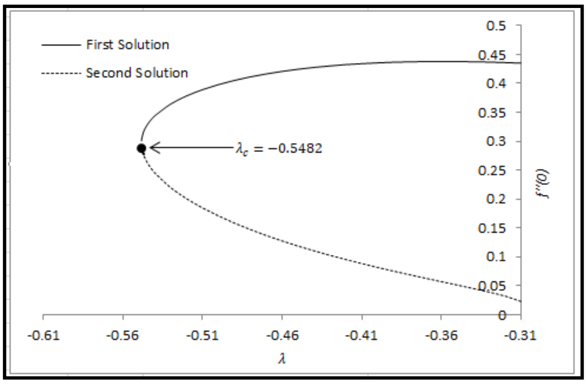

Figure 1. Variation of $f''(0)$ for Cu with $\varphi = 0$ and $R = 0.2$

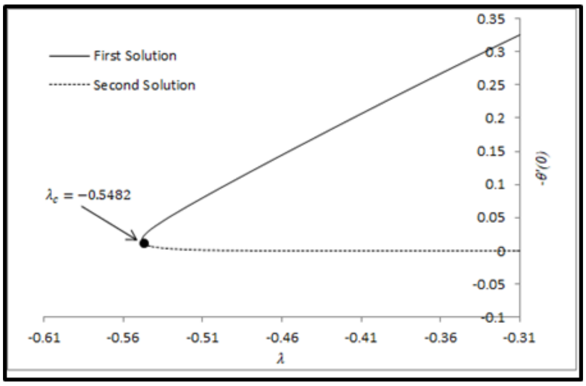

Figure 2. Variation of $-\theta'(0)$ for Cu with $\varphi = 0$ and $R = 0.2$

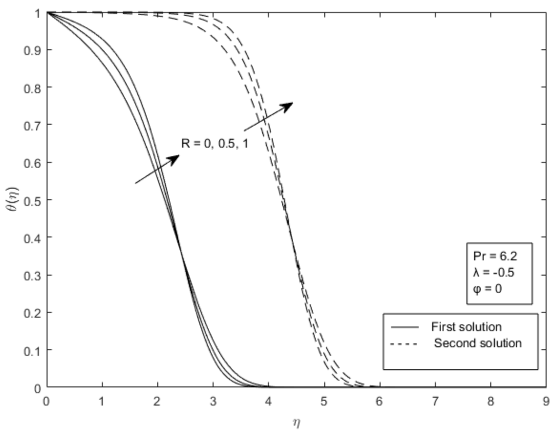

Figure 3. Temperature profiles for Cu with different values of R

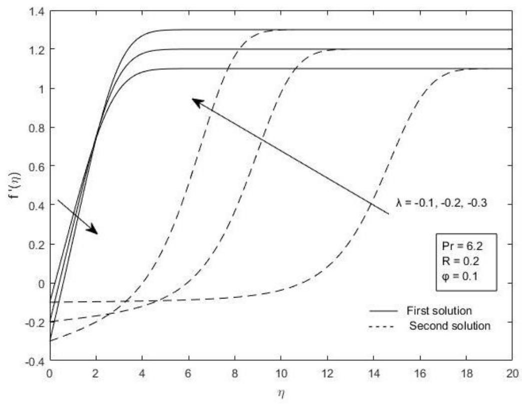

Figure 4. Velocity profiles for Cu with different values of λ

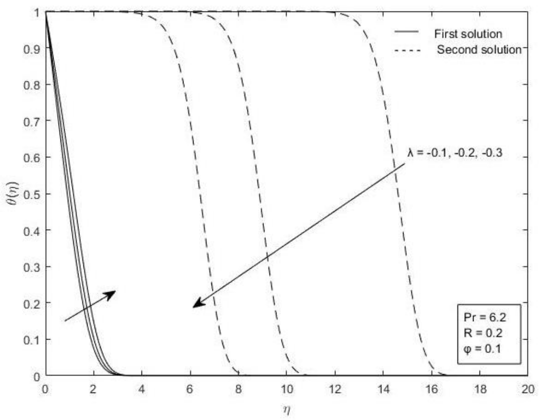

Figure 5. Temperature profiles for Cu with different values of λ

Since the non-unique solution present in this study, it is our determination to show that first solution is stable and physically realizable while the second solution is not. The stability of the flow can be tested by looking at thepolarity of the smallest eigenvalue, γ itself. The solution is unstable if the value of the smallest eigenvalue, γ is negative while it is stable if the smallest eigenvalue, γ is positive.

Table 4 presents the smallest eigenvalue, γ at selected values of λ. From Table 4, we found that first solution have positive value while second solution is negative value. Hence, we are sure that first solution is stable and realizable physically while second solution is vice versa. Besides, from Table 4, it clearly shown that as the values of λ are approaching λ_c, the smallest eigenvalue, γ tends to zero either from the positive side or negative side.

Table 4. Smallest eigenvalues γ at selected values of λ for different nanoparticle

Nanoparticle	λ	First solution	Second solution
Cu	-0.5482	0.0067	-0.0065
	-0.53	0.0621	-0.0497
	-0.52	0.0788	-0.0597
	-0.51	0.0931	-0.0672
Al_2O_3	-0.5482	0.0067	-0.0065
	-0.53	0.0621	-0.0497
	-0.52	0.0788	-0.0597
	-0.51	0.0931	-0.0672
TiO_3	-0.5482	0.0067	-0.0065
	-0.53	0.0621	-0.0497
	-0.52	0.0788	-0.0597
	-0.51	0.0931	-0.0672

V. CONCLUSION

The study of a stability analysis on laminar boundary layer flow saturated by nanofluids that passing a moving plate with the presence of thermal radiation has been numerically analyzed and discussed in details. Numerical findings from the transformation of partial differential boundary layer equations to a structure of ordinary differential equations have been obtained and well presented in the form of tables and figures. Dual solutions are found to be exist for certain ranges values of velocity ratio parameter $(\lambda > \lambda_c)$. Due to the possessing of dual solutions, an analysis of stability is conducted to identify which solution is stable between these two. It has been observed that the first solution is physically realizable and stable, while the second solution is not.

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VII. REFERENCES

Bachok, N., Ishak, A. & Pop, I. (2012). Flow and heat transfer characteristics on a moving plate in a nanofluid. International Journal of Heat and Mass Transfer, 55, pp. 642-648.

Choi, S. U. S. (1995). Enhancing thermal conducting of fluids with nanoparticles. International Mec-hanical Engineering Congress and Expo, 231, pp. 99-105.

Das, S. K., Choi, S. U. S. & Patel. H. E. (2006). Heat transfer in nanofluids-a review. Heat Transfer Engineering, 27 (10), pp. 3-19.

Gaffar, S. A., Prasad, V. R. & Reddy, E. K. (2017). Computational study of jeffrey's non-newtonian fluid past a semi-infinite vertical plate with thermal radiation and heat generation/absorption. Ain Shams Engineering Journal, 8, pp. 277-294.

Harris, S. D., Ingham, D. B. & Pop, I. (2009). Mixed convection boundary-layer flow near the stagnation point on a vertical surface in a porous medium: Brinkman model with slip. Transport Porous Media, 77, pp. 267-285.

Ishak, A. (2014). Flow and heat transfer over a shrinking sheet: A stability analysis. International Journal of Mechanical Industry Science Engineering, 8 (5), pp. 850-854.

Junoh, M. M, Ali, F. M., Arifin, N.M. &Bachok, N. (2018). Dual solutions in magnetohydrodynamic (MHD) flow on a nonlinear porous shrinking sheet: A stability analysis. In AIP Conference Proceedings, 1974, 020083, AIP Publishing.

Khan, M. S., Karim, I., Ali, L. E. & Islam, A. (2012). Unsteady MHD free convection boundary layer flow of a nanofluid along a stretching sheet with thermal radiation and viscous dissipation effects. International Nano Letters, 2 (24).

Khan, S., Karim, I., Islam, M.S. & Wahiduzzaman M. (2014). MHD boundary layer radiative, heat generating and chemical reacting flow past a wedge moving in a nanofluid. Nano Convergence, 1 (20).

Merkin, J.H. (1985). On dual solutions occurring in mixed convection in a porous medium. Journal of Engineering Mathematics, 20, pp. 171-179.

Oztop, H. F. & Abu-Nada (2008). Numerical study of natural convection in partially heated rectangular enclosures filled with nanofluids. International Journal Heat Fluid Flow, 29, pp. 1326-1336.

Sharma, R., Ishak, A. & Pop, I. (2014). Stability analysis of magnetohydrodynamic stagnation-point flow towards a stretching/shrinking sheet. Computers and Fluids, 102, pp. 94-98.

Weidman, P. D., Kubitschek, D. G. & Davis, A. M. J. (2006). The effect of transpiration on selft-similar boundary layer flow over moving surfaces. International Journal Engineering Science, 44, pp. 730-737.

Wong, K.V. & Leon, D.D. (2010). Applications of nanofluids: current and future. Advances in Mechanical Engineering, 2010, 519659.