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Graph Related to Cubed Commutativity Degree

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Let G be a finite group and $T^3(G)$ be the set of third power of commuting element in G *i.e* $T^3(G)=\left\{x\in G \mid (xg)^3=(gx)^3\right\}$. We define a graph, Γ with the vertex set $G\setminus T^3(G)$ in which two vertices x and y are joined by an edge (connected) if $(xy)^3\neq(yx)^3$. This graph is called as graph related to cubed commutativity degree. In this research, we study this graph and find a characterization of finite groups in term of this graph.

Keywords: graph, cubed commutativity degree

I. INTRODUCTION

Recently there are tremendous research to construct a graph by a group or semigroup. Since algebraic graph theory has close links with group theory, algebraic methods are applied to problem about graphs or vice versa. A new graph called graph related to conjugacy class was introduced by (Bertram et al., 1990*)* by considering its vertices are non-central conjugacy classes, where two vertices are adjacent if the cardinalities are not coprime. As a consequence, numerous works have been done on this graph and many results have been achieved. See (Bianchi et al., 1992; Chillag et al., 1993; You et al., (2005); Moreto, et al., 2005) for more details.

In 2012, (Erfanian & Tolue, 2012) introduced the relative non nil-n graph, $\Gamma^{(n)}_{H,G}$ of a finite group G. It is a graph with vertex set $G\setminus C^{(n)}_G(H)$ and two distinct vertices x and y are adjacent if at least one of them belongs to H and $[x,y]\notin Z_{n-1}(G)$, where the subgroup $C^{(n)}_G(H)$ contains $g\in G$ such that $[g,h]\in Z_{n-1}(G)$ for all $h\in H$. They proved that two n-isoclinic groups which are not nil-n groups have isomorphic graphs under special conditions. In the same year, (Erfanian & Tolue, 2012) introduced a new graph which is called a conjugate graph. The vertices of this graph are non-central elements of a finite non-abelian group. Two vertices of this graph are adjacent if they are conjugate. Moreover, (Bianchi et al. 2012) studied the regularity of the graph related to conjugacy class and provided some results.

Furthermore, the orbit graph is a result of generalization of conjugate graph. This graph was firstly introduced by (Omer et al., 2014). The number of vertices of an orbit graph is $|V(\Gamma^{\Omega}_G)|=|\Omega|-|A|$ where Ω is a disjoint union of distinct orbits under the action of G on the set Ω, while $A=\{v\in\Omega \mid vg=gv, g\in G\}$. Two vertices of this graph are linked by an edge if and only if there exists $g\in G$ such that $gw_1=w_2$, where $w_1,w_2\in\Omega$. A year later, El-sanfaz and Sarmin found the generalized conjugacy classes graph of metacyclic 2-groups of positive type of nilpotency class at least three.

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II. METHODS

This section focuses on new graph which is related to the cubed commutativity degree, denoted by $\Gamma_3(G)$. Let $w(x,y)$ be a word and W be the variety of groups defined by the law $w(x,y)=1$. Suppose that G is a group which is not in W. Let $T(G)=\{x\in G \mid w(x,g)=w(g,x)=1,$ for all $g\in G\}$. A graph, Γ is defined as follows;

Definition 1. *Thevertex set $V=V(\Gamma)$ of Γ is the set $G\setminus T(G)$ and two vertices $x,y\in V$ are joined by an edge if $w(x,y)\neq 1$.*

Note that to avoid isolated vertices, the vertex set is taken as the elements of G outside $T(G)$. If $w(x,y)=[x,y]=x^{-1}y^{-1}xy$, the graph $\Gamma(G)$ is the non-commuting graph investigated in (Abdollahi et al., 2006). Let *n* be a positive integer. One can investigate the graph of a group using the word $w(x,y)=[x,y^n]$. The variety of groups defined by the word $[x,y^n]$ coincides with the variety of groups defined by the word $(xy)^n(yx)^{-n}$. So, one can also investigate the graph of a group using the word $w(x,y)=(xy)^n(yx)^{-n}$. Therefore, a special word has been considered in this section.

Now, suppose that G is a group such that $T^n(G)$ is a proper subgroup. The vertex set $V=V(\Gamma)$ of the graph $\Gamma:=\Gamma(G)$ is the set $G\setminus T^n(G)$ and two elements $x,y\in V$ are joined by an edge if $(xy)^n\neq(yx)^n$.

The following is the definition of graph related to the cubed commutativity degree.

Definition 2.*Let G be a finite group and $T^3(G)$ be the set of third power of commuting elements in G, i.e $T^3(G)=\{x\in G \mid (xg)^3=(gx)^3\}$. Then the graph related to the cubed commutativity degree, $\Gamma_3(G)$ is defined as a graph whose vertices are non central elements in G but not in $T^3(G)$, that is $V(\Gamma_3(G))=G\setminus T^3(G)$. Note that $V:=V(\Gamma_3(G))=\{g\in G \mid (xg)^3\neq(gx)^3 \text{ for some } x\in G\}$ and two vertices are connected if $(xy)^3\neq(yx)^3$.*

The following example is given to illustrate the definition of graph related to cubed commutativity degree.

Example 1. *Let $G=D_5$, since $T^3(G)=\bigcap_{x\in G}T_G^3(x)$, then $T^3(D_5)=e$. Therefore, $V(\Gamma_3(D_5))=G\setminus\{e\}$ i.e $V(\Gamma_3(D_5))=\{a,a^2,a^3,a^4,b,ab,a^2b,a^3b,a^4b\}$. By using definition of graph related to the cubed commutativity degree, two elements are joined by an edge if $(xy)^3\neq(yx)^3$, thus the graph of $\Gamma_3(D_5)$ is produced and presented in the following.*

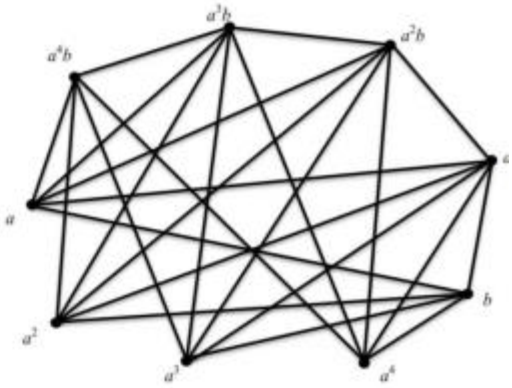

FIGURE 1: $\Gamma_3(D_5)$

III. RESULTS AND DISCUSSIONS

Throughout this research, G is assumed as finite group, $n=3$ and $\Gamma:=\Gamma_3(G)$. Some results have been investigated on this graph starting with the following lemma.

Lemma 1. *If G is not abelian, then Γ has no isolated vertex.*

Proof. Assume that $x \in V$ is an arbitrary vertex. Since $x \in G \setminus T^3(G)$, so $x \notin T^3(G)$ and therefore there exists an element $g \in G$ such that $(xg)^3 \neq (gx)^3$. Note that g is also a vertex since $g \in G$ and $g \notin T^3(G)$. Therefore, there is an edge between x and g. So, x is not an isolated vertex.

Remarks If G is abelian then $T^3(G) = G$. Next, groups with complete graphs are classified.

Theorem 1. *The graph Γ is a complete graph if and only if $H := C(G^3)$ is abelian of odd order with index 2, $x^2 = 1, h^x = h^{-1}$, for all $x \in G \setminus H$ and all $h \in H$.*

Proof. Let x be any element of G outside H. Then $G = H \bigcup xH$ and $V(\Gamma) = G \setminus H = xH$. Let xh_1 and xh_2 be two distinct vertices of Γ, where $h_1, h_2 \in H$. Then $(xh_1xh_2)^3 \neq (xh_2xh_1)$. Otherwise, $(h_1^x h_2)^3 = (h_2^x h_1)^3$. So $(h_1^{-1}h_2)^3 = (h_2^{-1}h_1)^3$ which implies that $(h_1^{-1}h_2)^6 = 1$, contradicting the hypothesis on the order of H. Therefore, xh_1 and xh_2 are adjacent and Γ is a complete graph.

Suppose that Γ is a complete graph. Suppose that $x \neq x^{-1}$ for some $x \in G \setminus H$. Then since $(x^{-1}x)^3 = (xx^{-1})^3$, x is not adjacent to x^{-1}, contradicting $\mathrm{diam}(\Gamma) = 1$. Hence, $x^2 = 1$ for all $x \in G \setminus H$. For all $h \in H$ and for all $xh \in G \setminus H$, we have $xh \in G \setminus H$ and so $(xh)^2 = 1$. Therefore, $h^x = h^{-1}$. To see that H is abelian, let $h_1, h_2 \in H$. Then $(h_2h_1)^{-1} = h_1^{-1}h_2^{-1} = h_1^x h_2^x = (h_1h_1)^x = (h_1h_2)^{-1}$, for all $x \in G \setminus H$, and so $h_1h_2 = h_2h_1$. Hence, H is abelian. Now, we prove that $|G/H| = 2$. Since G is not an elementary abelian 2-group *i.e* not all element have order two, there exists $h \in H$ such that $h^2 \neq 1$. If $|G/H| > 2$, then there exist $x_1x_2 \in G \setminus H$ such that $x_1x_2 = x_1^{-1}x_2 \notin H$. Hence, $h^{x_1x_2} = h^{-1}$ and so $h^{-1} = h^{x_1x_2} = (h^{x_1})^{x_2} = (h^{-1})^{x_2} = h$, which is contradiction. Hence, $|G/H| = 2$. Finally, we show that $|H|$ is odd. Suppose there exists a non-identity $h \in H$ such that $h^3 = 1$ and let $x \in G \setminus H$. Then $(xxh)^3 = (x^2h)^3 = h^3 = 1$ and $(xhx)^3 = (x^2h^x)^3 = (h^{-1})^3 = 1$. Hence, x and xh are not connected, contradicting to $\mathrm{diam}(\Gamma) = 1$. Therefore, $|H|$ is odd. The proof then follows.

Now, we prove some results when $[G : C(G^3)] = 2$.

Lemma 2. *Suppose that $H := C(G^3)$ has index 2 and let $x \in G \setminus H$ and $h \in H$ such that $h^3 \in C(x)$. Then $h^3 \in Z(G)$. Therefore, $A := \{h \in H \mid h^3 \notin Z(G)\}$ is non-empty, and for all $h \in H$, x and hx (as well as x and xh)are adjacent if and only if $h \in A$. Also, the graph Γ is $|A|$- regular.*

Proof. Let $g \in G$. It is clear that if $g \in H$, then $gh^3 = h^3g$. Suppose that $g = xh_1$ for some $h_1 \in H$. Thus, $gh^3 = (xh_1)h^3 = x(h_1h^3) = x(h^3h_1) = h^3(xh_1) = h^3g$, and therefore $h^3 \in Z(G)$, which proves the first assertion. Now note that $(xhx)^3 = (hxx)^3$ if and only if $xhx^2hx^2hx = hx^2hx^2hx^2$ if and only if $x^5h^3x = x^4h^3x^2$ if and only if $xh^3 = h^3x$ if and only if $h^3 \in C(x)$. Thus, x and hx are adjacent if and only if $h \in A$. To see that A is non-empty, note that since $x \in H$ and $H = C(G^3) = T^3(G)$, there exists $g \in G$ such that $(xg)^3 \neq (gx)^3$. Hence, Γ has at least one edge. Suppose that (x,hx), for some $h\in H$ is an edge. Then by the above observation, $h^3\notin Z(G)$ and so $A\neq\phi$. Since x and hx are adjacent if and only if $h\in A$, thus all vertices of the graph have degree of $|A|$. Therefore, the graph is $|A|$-*regular*.

Now, we can determine the girth of Γ.

Theorem 2. *Suppose that* $H:=C(G^3)$ *has index 2. Then the graph* Γ *has girth 3 or 4. Also the girth is 3 if and only if either*

i. *There exists* $h\in H$ *such that* $h^6\notin Z(G)$ *or*

ii. $\{h^6 \mid h\in H\}\subseteq Z(G)$ *and there exist* $h,k\in A$ *such that* $kh^{-1}\in A$*, where* $A=\{h\in H \mid h^3\notin Z(G)\}$*, and* $k^x\neq h$*, for some* $x\in G\setminus H$.

Proof. Let x be any element of G outside H. First, we note that if $h\in H$, then xh and hx are not adjacent if and only if $h^6\in Z(G)$. In fact, if xh and hx are not adjacent, then $(xhhx)^3=(hxxh)^3$. Since $hx^2=x^2h$, we have $(xh^2x)^3=(hx^2h)^3$ implies $(xh^2x)^3=(h^2x^2)^3$ and so $xh^2xxh^2xxh^2x=h^2x^2h^2x^2h^2x^2$. Hence $xh^6x^5=h^6x^6$ and $xh^6=h^6x$. Since for all $u\in H$ we have $uh^6=h^6u$, it follows that $h^6\in Z(G)$. This argument also shows that if $h^6\in Z(G)$, then xh and hx are not adjacent.

Suppose that condition (i) holds, that is $h^6\in Z(G)$, for some $h\in H$. By Lemma 2, (x,xh) and (x,hx) are edges of Γ. Since $h^6\notin Z(G)$, by observation, xh and hx are adjacent. Thus (x,xh,hx) form a triangle and so $\text{girth}(\Gamma)=3$.

Now suppose that condition (ii) holds, that is $\{h^6 \mid h\in H\}\subseteq Z(G)$ and there exist $h,k\in H$ such that $\{h^3,k^3,(kh^{-1})^3\}\bigcap Z(G)=\phi$. By Lemma 2, (x,hx) and (x,kx) are edges on Γ. Also, by Lemma 2, $(hx,kh^{-1}(hx))=(hx,kx)$ is an edge of Γ. Thus, there is a triangle with vertices x, hx and kx. Hence, $\text{girth}(\Gamma)=3$.

Now suppose that the girth of Γ is 3. If (x,xh,hx) is a triangle, then $h^6\notin Z(G)$ and condition (i) holds. So suppose that $\{h^6 \mid h\in H\}\subseteq Z(G)$. Given a triangle with vertices h_1x, h_2x and h_3x, set $z=h_1x$ and setting $k_2=h_2h_1^{-1}$ and $k_3=h_3h_1^{-1}$ the vertices of the triangle become z, k_2z and k_3z. So it might be assumed that any triangle has vertices x, hx and kx, where $x\notin H$, $h,k\in H$ and by Lemma 2, $h,k\in A$. Viewing the edge (hx,kx) as $(hx,kh^{-1}(hx))$, there is also $kh^{-1}\in A$. As, in the case, (hx,xh) is not as edge of Γ, it must has $kx\neq xh$ so $k^x\neq h$. Therefore, the condition (ii) holds.

By Lemma 2, there exists a non-identity element $h\in H$ such that $h^3\notin Z(G)$. Suppose that the conditions (i) and (ii) do not hold. Then $h^6\in Z(G)$. Since $h^3\notin Z(G)$, by Lemma 2, $(x,hx),(x,h^{-1x}),(hx,h(hx))=(hx,h^2x)$ are edges of Γ. Now, if $h^3=1$, then since $h^6\in Z(G)$ it has been concluded that $h\in Z(G)$, contradicting $h^3\notin Z(G)$. Thus, $h^3\neq 1$ and, by Lemma 2, $(h^{-1}x,h^{-1}(h^3x))=(h^{-1}x,h^2x)$ is an edge of Γ. Hence, $\{x,hx,h^2x,h^{-1}x\}$ form a square. Therefore, $\text{girth}(\Gamma)=4$.

IV. SUMMARY

In this paper, new graph has been introduced which is the graph related to cubed commutativity degree. It is proven that the graph is complete if and only if its subgroup is abelian of odd order with index two.

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