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Numerical Investigation to Fuzzy Volterra Integro-Differential Equations via Residual Power Series Method

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In this paper, a study of a numerical approximate solution to fuzzy Volterra integro-differential equations is presented under strongly generalised differentiability by applying an influent effective technique, called the Residual Power Series (RPS) method. The solution approach can be expressed on Taylor's series formula in terms of elementary σ-level representation, whereas the coefficients can be computed by utilising its residual functions. Furthermore, a numerical computational example is given to test and validate the proposed method. The results reached show several features concerning the RPS method such as potentiality, generality and superiority to handle many problems arising in physics and engineering.

Keywords: fuzzy IVPs; integro-differential equation; residual power series method; Hukuhara differentiability

I. INTRODUCTION

In recent years, the topic of fuzzy integro-differential equations (IDEs) has gained more attention by many scholars for its essential role in modelling many phenomena that arise in various applications in physics and engineering (Gumah *et al.*, 2016; Abu Arqub *et al.*, 2016; Arqub *et al.*, 2017). Several scholars in scientific areas widely use the fuzzy IDEs to understand the structure and predict the behaviour of solutions to the issues under study. Typically, during the formulation of many problems, the resulting information will be subject to uncertainty for various reasons such as errors in the measurement process, inaccurate prediction, collection of erroneous data, as well as bad estimation when determining the initial guesses. Therefore, effective mathematical tools are needed to understand this uncertainty. Consequently, the theory of fuzzy set is a powerful tool for handling and modelling such issues under uncertainty. For other numerical techniques about integro-differential equations, we refer to (Abu Arqub & Al-Smadi, 2018; Al-Smadi & Arqub, 2019; Saadeh *et al.*, 2016; Abu Arqub & Al-Smadi, 2018).

On another aspect, investigations of numerical solutions for fuzzy IDEs are missing and rare. In (Abbasbandy & Hashemi, 2012; Abbasbandy & Hashemi, 2010), the authors relied on the method of homotopy analysis and the variational iteration method to find a numerical solution of such equations. The reproducing kernel algorithm was applied for solving a class of fuzzy Fredholm–Volterra integro-differential equations (Abu Arqub, 2017). Haar wavelet method was used to solve a class of fuzzy Volterra integro-differential equations (Altawallbeh *et al.*, 2018). In the current paper, we extend the application of the residual power series (RPS) method to investigate the approximate analytic solutions of fuzzy Volterra IDEs under generalised H-differentiability (Abu-Gdairi *et al.*, 2015; Al-Smadi *et al.*, 2017; Al-smadi, 2019; Abu Arqub *et al.*, 2018; Al-Smadi, 2018; Freihet *et al.*, 2019).

The skeleton of this article is structured as follows: In section 2, some basic definitions and preliminary facts related to fuzzy calculus are given. The fuzzy Volterra integro-differential equations are discussed in section 3. In section 4, a description of the RPS method is presented. One numerical example is given to illustrate the RPS method. Finally, a summary of this paper has been provided.

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II. PRELIMINARIES

The material in this section is fundamental of some concepts. For convenience, the necessary definitions and theories used in the theory of fuzzy calculus are briefly given.

Definition 1. (Kaleva, 1987) A non-empty fuzzy set v in a universe of discourse X is described by a membership function $\mu_v: X \to [0,1]$, which associates with every point in X a real number in $[0,1]$. $\mu_v(x)$ is interpreted as the grade of membership of an element x in the fuzzy set v for each $x \in X$.

Definition 2. (Kaleva, 1987) Let $\mathbb{R}$ stands to the set of all real numbers and v is a non-empty set in $\mathbb{R}$. Then one says v is a fuzzy number if it holds the following requirement:

1- v is normal if there exists $x_0 \in \mathbb{R}$ for which $v(x_0) = 1$.
2- v is convex, for each $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$ which holds that $v(\lambda x_1 + (1-\lambda)x_2) \geq \min\left(v(x_1), v(x_2)\right)$.
3- v is an upper semi-continuous, for each $x_0 \in \mathbb{R}$, $v(x_0) \geq \lim_{x \to x_0^+} v(x)$ and $v(x_0) \geq \lim_{x \to x_0^-} v(x)$.
4- The closure of $\operatorname{supp}(v)$ is compact.

For each $\sigma \in (0,1]$, set $[v]^\sigma = \{x \in \mathbb{R}: v(x) \geq \sigma\}$ and $[v]^0 = \overline{\{x \in \mathbb{R} : v(x) > 0\}}$, where $\overline{\{\cdot\}}$ denote the closure of $\{\cdot\}$. Then, it can be easily proved that v is a fuzzy number if and only if $[v]^\sigma$ is compact and convex set on $\mathbb{R}$ for all $\sigma \in [0,1]$ and $[v]^1 \neq \emptyset$ (Goetschel & Voxman, 1986). Further, let v be a fuzzy number, then $[v]^\sigma = [v_1(\sigma), v_2(\sigma)]$. That is, $[v]^\sigma$ referred to the σ-cut representation or parametric form of a fuzzy number v. Where $v_1(\sigma) = \min\{x : x \in [v]^\sigma\}$ and $v_2(\sigma) = \max\{x : x \in [v]^\sigma\}$ for each $\sigma \in [0,1]$.

Theorem 3. (Goetschel & Voxman, 1986) Suppose that $v_1, v_2: [0,1] \to \mathbb{R}$ which satisfies the following conditions;

1- v_1 is a bounded increasing function, and v_2 is a bounded decreasing function with $v_1(1) \leq v_2(1)$;
2- v_1 and v_2 are left-hand continuous functions at $\sigma = k$, for all $k \in (0,1]$,
3. v_1 and v_2 are right-hand continuous functions at $\sigma = 0$.

Then, $v: \mathbb{R} \to [0,1]$ defined by $v(x) = \sup\{\sigma : v_1(\sigma) \leq x \leq v_2(\sigma)\}$ is a fuzzy number and the parametric form is $[v_1(\sigma), v_2(\sigma)]$. Otherwise, the functions v_1 and v_2 satisfy the conditions above if $v: \mathbb{R} \to [0,1]$ is a fuzzy number with parametrisation $[v_2(\sigma), v_1(\sigma)]$.

Definition 4. (Puri & Ralescu, 1983) For arbitrary fuzzy numbers v and w on $\mathbb{R}_F$ we define the Hausdorff distance by the mapping $d: \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$ such that

$$d(v, w) = \sup_{0 \leq \sigma \leq 1} \max\{|v_1(\sigma) - w_1(\sigma)|, |v_2(\sigma) - w_2(\sigma)|\},$$

Definition 5. (Puri & Ralescu, 1983) Let $f: [a,b] \to \mathbb{R}_F$ be a fuzzy valued function. For fixed $x_0 \in [a,b]$ and $\epsilon > 0$, there is $\delta > 0$ with $|x - x_0| < \delta$ this implies $d(f(x), f(x_0)) < \epsilon$, then we say that f is continuous at x_0.

Definition 6. (Friedman *et al.*, 1999) Suppose that $f: [a,b] \to \mathbb{R}_F$ be a fuzzy valued function. For each partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ and $\zeta_i \in [x_{i-1}, x_i]$, $1 \leq i \leq n$. Assume that $R_P = \sum_{i=1}^{n} x_i(\zeta_i)(x_i - x_{i-1})$ and $\Delta = \max_{1 \leq i \leq n} |x_{i-1}, x_i|$. The definite integral of $f(x)$ over $[a,b]$ is $\int_a^b f(x)dx = \lim_{\Delta \to 0} R_P$ such that the limit exists in $(\mathbb{R}_F, d)$. If the fuzzy function $f(x)$ is continuous in the metric d, its definite integral exists (Goetschel & Voxman, 1986). Moreover,

$$\left(\int_a^b f(x)dx\right)_{1\sigma} = \int_a^b f_{1\sigma}(x)dx,$$

and

$$\left(\int_a^b f(x)dx\right)_{2\sigma} = \int_a^b f_{2\sigma}(x)dx.$$

The fuzzy number A is called the Hukuhara difference or “H-difference” of $v, w \in \mathbb{R}_F$ and indicated by $v \ominus w$, which is mean $\forall\, v, w \in \mathbb{R}_F$, $\exists$ an element $A \in \mathbb{R}_F$ provided that $v = w + A$. As well we referred always for Hukuhara difference by the sign “$\ominus$” and let us mention that $v \ominus w \neq v + (-1)w$.

Definition 7. (Bede & Gal, 2005) Let $f: [a,b] \to \mathbb{R}_F$ and fixed $x_0 \in [a,b]$. We say that f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in \mathbb{R}_F$ such that either:

i- The H-differences $f(x_0 + h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0 - h)$ exist and

$$f'(x_0) = \lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(x_0 - h)}{h},$$

for all $h > 0$ sufficiently close to 0, where the limits in a metric d, ii-The H-differences $f(x_0) \ominus f(x_0 + h), f(x_0 - h) \ominus f(x_0)$ exist and

$f'(x_0) = \lim_{h\to 0^+} \frac{f(x_0) \ominus f(x_0+h)}{-h} = \lim_{h\to 0^+} \frac{f(x_0-h)\ominus f(x_0)}{-h}$,

for all $h > 0$ sufficiently close to 0, where the limits in a metric d.

Theorem 8. (Chalco-Cano & Román-Flores, 2008) Assume that v: $[a, b] \to \mathbb{R}_F$, $[f(x)]^\sigma = [f_{1\sigma}(x), f_{2\sigma}(x)]$ for each $\sigma \in [0,1]$, then:

1- If f is (1)-differentiable, then $f_{1\sigma}$ and $f_{2\sigma}$ are differentiable functions and $[D_1^1 f(x)]^\sigma = [f'_{1\sigma}(x)\ ,\ f'_{2\sigma}(x)]$,

2- If f is (2)-differentiable, then $f_{1\sigma}$ and $f_{2\sigma}$ are differentiable functions and $[D_2^1 f(x)]^\sigma = [f'_{2\sigma}(x)\ ,\ f'_{1\sigma}(x)]$.

III. FUZZY VOLTERRA INTEGRODIFFERENTIL EQUATIONS

This section aims to provide and discuss the fuzzy approximate solution of the following fuzzy Volterra integro-differential equations with the fuzzy initial conditions under the concept of strongly generalised differentiability

$$u'(x) = \varphi(x) + \lambda \int_0^x g(x,t)u(t)dt, \qquad (1)$$

subject to the initial condition

$$u(0) = u_0, \qquad (2)$$

where λ is a positive parameter, $g(x,t)$ is a function, called an arbitrary kernel function and φ is a continuous function of x. To solve Eq. (1) and (2), we write the given fuzzy equation in the parametric forms as follows:

$$\begin{aligned} u'_{1\sigma}(x) &= \varphi_{1\sigma}(x) + \lambda \int_0^x G_{1\sigma}(x,t,u(t))dt, \\ u'_{2\sigma}(x) &= \varphi_{2\sigma}(x) + \lambda \int_0^x G_{2\sigma}(x,t,u(t))dt, \end{aligned} \qquad (3)$$

with $u_{1\sigma}(0) = u_{01\sigma}$ and $u_{2\sigma}(0) = u_{02\sigma}$,

Where,

$$G_{1\sigma}(x,t,u(t)) = \begin{cases} g(x,t)\,u_{1\sigma}(t), & g(x,t) \ge 0, \\ g(x,t)\,u_{2\sigma}(t), & g(x,t) < 0, \end{cases}$$

and

$$G_{2\sigma}(x,t,u(t)) = \begin{cases} g(x,t)\,u_{2\sigma}(t), & g(x,t) \ge 0, \\ g(x,t)\,u_{1\sigma}(t), & g(x,t) < 0. \end{cases}$$

IV. DESCRIPTION OF THE RPS METHOD

In this section, we describe the RPS scheme to provide the approximate solution for Eq. (1) and (2). To do so, we assume that the solution of Eq. (3) has the following power series expansion about the initial point $x_0 = 0$:

$$\begin{aligned} u_{1\sigma}(x) &= \sum_{n=0}^{\infty} a_n x^n, \\ u_{2\sigma}(x) &= \sum_{n=0}^{\infty} b_n x^n. \end{aligned} \qquad (4)$$

Applying the initial conditions of Eq. (3) in the series solutions in Eq. (4), we get that $u_{1\sigma}(0) = u_{01\sigma} = a_0$ and $u_{2\sigma}(0) = u_{02\sigma} = b_0$, and hence we can approximate the solutions $u_{1\sigma}(x)$ and $u_{2\sigma}(x)$ by the m^{th}-truncated series as:

$$\begin{aligned} u_{m,1\sigma}(x) &= a_0 + \sum_{n=1}^{m} a_n x^n, \\ u_{m,2\sigma}(x) &= b_0 + \sum_{n=1}^{m} b_n x^n. \end{aligned} \qquad (5)$$

According to the RPS approach (Moaddy *et al.*, 2015; Gumah *et al.*, 2018; Komashynska *et al.*, 2016; Hasan *et al.*, 2019; Komashynska *et al.*, 2016) in finding the unknown constants a_n and $b_n, n = 1,2,\dots,m$, we define the m^{th}-residual functions as the following:

$$\begin{aligned} Res_{m,1\sigma}(x) &= u'_{m,1\sigma}(x) - \varphi_{1\sigma}(x) \\ &\quad - \lambda \int_0^x g(x,t)\, u_{m,1\sigma}(t)dt, \\ Res_{m,2\sigma}(x) &= u'_{m,2\sigma}(x) - \varphi_{2\sigma}(x) \\ &\quad - \lambda \int_0^x g(x,t)\, u_{m,2\sigma}(t)dt. \end{aligned} \qquad (6)$$

Now, to obtain the unknown constants a_1 and b_1, substitute $u_{1,1\sigma}(x) = a_0 + a_1 x$ and $u_{1,2\sigma}(x) = b_0 + b_1 x$, into the first residual equation as follows

$$\begin{aligned} Res_{1,1\sigma}(x) &= a_1 - \varphi_{1\sigma}(x) - \lambda \int_0^x g(x,t)\,(a_0 + a_1 t)dt, \\ Res_{1,2\sigma}(x) &= b_1 - \varphi_{2\sigma}(x) - \lambda \int_0^x g(x,t)\,(b_0 + b_1 t)dt \end{aligned} \qquad (7)$$

Then, by using the facts $\text{Res}_{1,1\sigma}(0) = \text{Res}_{1,2\sigma}(0) = 0$, it yields

$$\begin{aligned} a_1 &= \left(\frac{\varphi_{1\sigma}(x) + \lambda a_0 \int_0^x g(x,t)dt}{1 - \lambda \int_0^x t g(x,t)dt}\right)_{x=0}, \\ b_1 &= \left(\frac{\varphi_{2\sigma}(x) + \lambda b_0 \int_0^x g(x,t)dt}{1 - \lambda \int_0^x t g(x,t)dt}\right)_{x=0}. \end{aligned} \qquad (8)$$

Again, to find a_2 and b_2, substitute $u_{2,1\sigma}(x) = a_0 + a_1 x + a_2 x^2$ and $u_{2,2\sigma}(x) = b_0 + b_1 x + b_2 x^2$ into the second residual equation as

$$Res_{2,1\sigma}(x) = (a_1 + 2a_2x) - \varphi_{1\sigma}(x) - \lambda \int_0^x g(x,t)\,(a_0 + a_1t + a_2t^2)dt, \quad (9)$$

$$Res_{2,2\sigma}(x) = (b_1 + 2b_2x) - \varphi_{2\sigma}(x) - \lambda \int_0^x g(x,t)\,(b_0 + b_1t + b_2t^2)dt.$$

Using the fact $\frac{d}{dx}\text{Res}_{2,1\sigma}(0) = \frac{d}{dx}\text{Res}_{2,2\sigma}(0) = 0$, we get that

$$a_2 = \left(\frac{\varphi'_{1\sigma}(x)+\lambda a_0\int_0^x \frac{\partial}{\partial x}g(x,t)dt+\lambda a_1\int_0^x \frac{\partial}{\partial x}g(x,t)\,t dt}{2-\lambda\int_0^x \frac{\partial}{\partial x}g(x,t)\,t^2\,dt}\right)_{x=0},$$

$$b_2 = \left(\frac{\varphi'_{2\sigma}(x)+\lambda b_0\int_0^x \frac{\partial}{\partial x}g(x,t)dt+\lambda b_1\int_0^x \frac{\partial}{\partial x}g(x,t)\,t dt}{2-\lambda\int_0^x \frac{\partial}{\partial x}g(x,t)\,t^2\,dt}\right)_{x=0}. \quad (10)$$

Likewise, for $m = 3$, and by using the facts $\frac{d^2}{dx^2}\text{Res}_{3,1\sigma}(0) = \frac{d^2}{dx^2}\text{Res}_{3,2\sigma}(0) = 0$, we can find a_3 and b_3 as

$$a_3 = \left(\frac{1}{6-\lambda\int_0^x \frac{\partial^2}{\partial x^2}g(x,t)\,t^3 dt}\left(\varphi''_{1\sigma}(x) + \lambda a_0\int_0^x g(x,t)dt + \lambda a_1\int_0^x \frac{\partial^2}{\partial x^2}g(x,t)\,t dt + \lambda a_2\int_0^x \frac{\partial^2}{\partial x^2}g(x,t)\,t^2\,dt\right)\right)_{x=0},$$

$$b_3 = \left(\frac{1}{6-\lambda\int_0^x \frac{\partial^2}{\partial x^2}g(x,t)\,t^3 dt}\left(\varphi''_{2\sigma}(x) + \lambda b_0\int_0^x g(x,t)dt + \lambda b_1\int_0^x \frac{\partial^2}{\partial x^2}g(x,t)\,t dt + \lambda b_2\int_0^x \frac{\partial^2}{\partial x^2}g(x,t)\,t^2\,dt\right)\right)_{x=0}.$$

V. ILLUSTRATIVE EXAMPLE

In the current section, we illustrate the efficiency and applicably of the proposed method by providing the numerical solutions of the following fuzzy Volterra integro-differential equation

$$u'(x) = \alpha(e^x(2-x)-1) + \int_0^x tu(t)\,dt\,,\ x \in [0,1]\,,\ 0 < t \le x, \quad (11)$$

with the fuzzy initial conditions

$$u(0) = \alpha,$$

where the σ-cut representation of α is $[0.5 + 0.5\sigma, 2-\sigma]$, $0 \le \sigma \le 1$.

The exact solution is given by $u(x) = [0.5 + 0.5\sigma, 2-\sigma]e^t$.

Thus, Eq. (11) can be written in the parametric forms as follows:

$$u'_{1\sigma}(x) = (0.5+0.5\sigma)(e^t(2-t)-1)\int_0^x tu_{1\sigma}(t)\,dt, \quad (12)$$

$$u'_{2\sigma}(x) = (2-\sigma)(e^t(2-t)-1) + \int_0^x tu_{2\sigma}(t)\,dt,$$

Subject to $u_{1\sigma}(0) = 0.5 + 0.5\sigma$ and $u_{2\sigma}(0) = 2-\sigma$.

To show the accuracy of the RPS technique, the exact and approximate solutions are compared in Table 1. The results obtained by the RPS method show that the exact solutions are in good agreement with approximate solutions at $\sigma = 1$ and $n = 8$.

Table 1: Numerical results at $\sigma = 1$ and $n = 8$.

x_i	Exact	Approximation	Absolute error
0.1	1.105170918	1.10517091807	2.88657×10^{-15}
0.3	1.349858807	1.34985880752	5.59139×10^{-11}
0.5	1.648721271	1.64872126503	5.66416×10^{-9}
0.7	2.013752707	2.01375258795	1.19513×10^{-7}
0.9	2.459603111	2.45960193894	1.17221×10^{-6}

To show the fuzzy behaviour of the proposed algorithm, the lower and upper approximate solutions are plotted in Figures 1, 2 and 3 for $t = 0, 0.25$ and 0.75 with $n = 5$, and $\sigma \in [0,1]$. From these figures, it can be noted that the solutions are represented in the shapes of symmetric triangular fuzzy along the direction of σ and with different values of t.

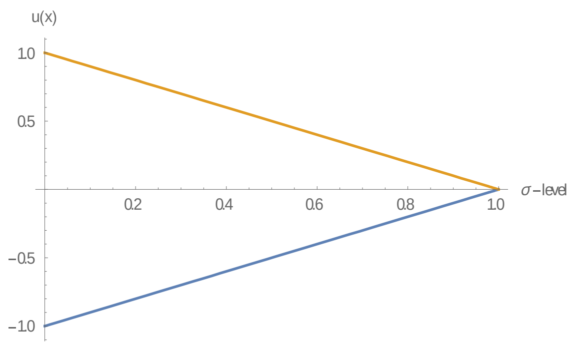

Figure 1: Solution plot for $\sigma \in [0,1]$, $t = 0$, $n = 5$.

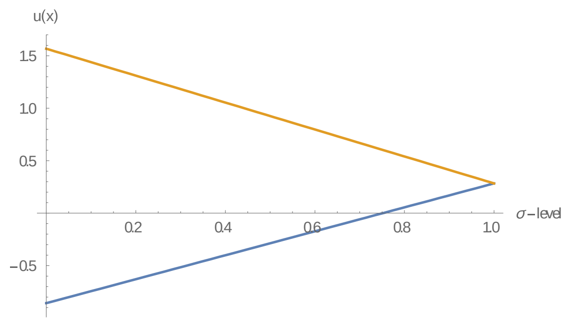

Figure 2: Solution plot for $\sigma \in [0,1]$, $t = 0.25$, $n = 5$.

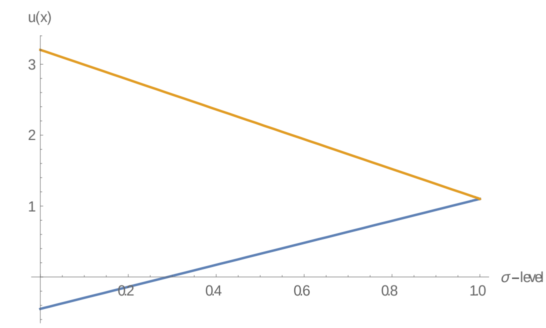

Figure 3: Solution plot for $\sigma \in [0,1]$, $t = 0.75$, $n = 5$.

VI. CONCLUSION

In this paper, the RPS scheme has been applied to provide the approximate solution for fuzzy Volterra IDEs with appropriate fuzzy initial conditions under strongly generalised H-differentiability. This method can be applied directly to the given problem by choosing an appropriate initial guess approximation. Numerical results have shown the performance and reliability of the present approach. The results indicate that the RPS method is very efficient and powerful in solving fuzzy differential equations with fewer calculations and time.

VII. ACKNOWLEDGEMENT

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