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Inclusion and Convolution Properties of Certain Subclasses of Analytic Functions Defined by Integral Operator

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In this paper, we investigate certain subclasses of analytic functions by means of integral operator connected with the incomplete beta function. A method of Noor integral is used to form a new integral operator. Various properties of these subclasses are considered. These include inclusion relation and convolution properties.

Keywords: Analytic function; integral operator; incomplete beta function; convolution; starlike function; convex function

I. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $0 \le \alpha < 1$, we denote by $S^*(\alpha), K(\alpha)$ the subclasses of $\mathcal{A}$ of all analytic functions that are starlike of order α and convex of order α respectively. In particular, the classes $S^*(0) = S^*, K(0) = K$.

Padamanabhan & Parvatham (1975) introduced a class $\mathcal{P}_k(\rho)$ of functions $\tau(z)$ which are analytic in $\mathbb{U}$, satisfying the condition $\tau(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re e\{\tau(z) - \rho\}}{1-\rho} \right| d\theta \le k\pi, \tag{2}$$

where $z = re^{i\theta}, k \ge 2,$ and $0 \le \rho < 1$. For $\rho = 0$ the class $\mathcal{P}_k(0) = \mathcal{P}_k$ was introduced by Pinchuk (1971). Also, we note that $\mathcal{P}_k(\rho) = \mathcal{P}(\rho)$ where $\mathcal{P}(\rho)$ is the classes of functions with real part greater than ρ and, $\mathcal{P}_2(0) = \mathcal{P}$, where $\mathcal{P}$ is the class of positive real part. From (2) we can easily deduce that

$$\tau(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1-2\rho) z e^{-it}}{1 - z e^{-it}} d\mu(t), \tag{3}$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$, such that

$$\int_0^{2\pi} d\mu(t) = 2 \quad and \quad \int_0^{2\pi} |d\mu(t)| \le k.$$

From (2) and (3) it can be seen that $\tau \in \mathcal{P}_k(\rho)$ if and only if there exist $\tau_1, \tau_2 \in \mathcal{P}(\rho)$ such that

$$\tau(z) = \left(\frac{k}{4} + \frac{1}{2}\right) \tau_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) \tau_2(z). \tag{4}$$

For more details in deriving (4), one can refer to Noor et al., (2012).

If $f(z)$ of the form (1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ are

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two functions in $\mathcal{A}$, then the Hadamard product (or convolution) $f * g$ is defined by

$$f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, (z \in \mathbb{U}). \tag{5}$$

For $a >$ and $b > 0$, we define the function $f(a,b)(z)$ by

$$f(a,b)(z) = \sum_{k=0}^{\infty} \frac{(b)_k}{(a)_k} z^{k+1} = \phi(b,a;z), \tag{6}$$

where $(\nu)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(k)} = \begin{cases} 1, & k=0, \nu \in \mathbb{C}\backslash\{0\}; \\ \nu(\nu+1)(\nu+2)....(\nu+k-1), & k \in \mathbb{N} = \{1,2,3,...\}, \end{cases} \tag{7}$$

and $\phi(b,a;z)$ is the incomplete beta function defined by $\phi(b,a;z) = z_2F_1(1,b;a,z)$, such that $z_2F_1(a,b;c,z)$ is the well-known Gaussian hypergeometric function given by the series

$$z_2F_1(a,b;c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}.$$

For $m, d \in \mathbb{N}_0 = \{0,1,2,...\}, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0$, and $\ell + d > 0$ the authors in (Oshah & Darus, 2014) have defined a function $\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}$ as follows:

$$\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}(z) = \sum_{k=1}^{\infty} \left[\frac{\nabla^k_{\lambda_2,\ell,d} + \ell\lambda_1(k-1)}{\nabla^k_{\lambda_2,\ell,d}}\right]^m z^k, \tag{8}$$

where

$$\nabla^k_{\lambda_2,\ell,d} = \ell(1+\lambda_2(k-1)) + d.$$

Now to get the integral operator, we follow methods by Noor (1999) and define a function$[\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}(z)]^{-1}$ in terms of Hadamard product (or convolution) by

$$\mathrm{M}^m_{\lambda_1,\lambda_2,\ell,d}(z) * [\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}(z)]^{-1} = \frac{z}{(1-z)^{\mu+1}},$$

$$(\mu > -1, z \in \mathbb{U}), \tag{9}$$

where

$$\frac{z}{(1-z)^{\mu+1}} = \sum_{k=1}^{\infty} \frac{(\mu+1)_{k-1}}{(1)_{k-1}} z^k, (z \in \mathbb{U}).$$

Then we have the integral operator $\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d} : \mathcal{A} \to \mathcal{A}$ as follows:

$$\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d} f(z) = [\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}(z)]^{-1} * f(z), \tag{10}$$

where,

$$(f \in \mathcal{A}, m, d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0)$$
$$(\ell \geq 0, \ell + d > 0, \mu > -1).$$

For a function $f(z) \in \mathcal{A}$ given by (1), it is easy to see from equation (10) that

$$\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d} f(z) =$$

$$z + \sum_{k=2}^{\infty} \left[\frac{\nabla^k_{\lambda_2,\ell,d}}{\nabla^k_{\lambda_2,\ell,d} + \ell\lambda_1(k-1)}\right]^m \frac{(\mu+1)_{k-1}}{(k-1)!} a_k z^k, \tag{11}$$

$m, d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, z \in \mathbb{U}$ and

$\ell + d > 0.$

Next, we define an operator

$\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,c) : \mathcal{A} \to \mathcal{A}$ by

(7) $$\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,c) f(z) = f(a,b)(z) * \mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d} f(z), \tag{12}$$

such that $\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d} f(z)$ is given by (10) and $f(a,b)(z)$ is
(8)
given by (6) with $m, d \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0$ and $\ell + d > 0$.

Note that $\mathcal{I}^{0,\mu}_{\lambda_1,\lambda_2,\ell,d}(1,1) f(z) = D^{\mu}$ and $\mathcal{I}^{0,0}_{\lambda_1,\lambda_2,\ell,d}(a,b) f(z) = \mathcal{L}(b,a) f(z)$ which are Ruscheweyh and Carlson-Shaffer derivative operators,
(9)
respectively (Ruscheweyh, 1982; Carlson & Shaffer, 1984). It is readily verified from (12) that

$$\mathcal{I}^{0,\mu}_{\lambda_1,\lambda_2,\ell,d}(1,\mu+1) f(z) = f(z),$$
$$\mathcal{I}^{0,1}_{\lambda_1,\lambda_2,\ell,d}(a,a) f(z) = z f'(z),$$

$$\mu\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z) = (\mu+1)\mathcal{I}^{m,\mu+1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z)$$
$$-z\left(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z)\right),$$
$$(a-1)\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a+1,b)f(z) = a\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z)$$
$$-z(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a+1,b)f(z))'.$$

For $k \geq 2, 0 \leq \gamma \leq 1$ and $0 \leq \rho < 1$, we introduce the following subclasses by using the integral operator $\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z)$,

$$\mathcal{Q}^{m,\mu}_k(a,b,\gamma,\rho) =$$
$$\{f \in \mathcal{A}:$$
$$\left(\frac{z(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))' + \gamma z^2(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))''}{(1-\gamma)(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z)) + \gamma z(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'}\right) \in \mathcal{P}_k(\rho), z \in \mathbb{U},$$
$$\mathcal{T}^{m,\mu}_k(a,b,\gamma,\rho) =$$
$$\left\{f \in \mathcal{A}: (\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))' + \gamma z(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'' \in \mathcal{P}_k(\rho), z \in \mathbb{U}\right\}.$$

Note that

(1) $\mathcal{Q}^{0,\mu}_k(\mu+1,1,\gamma,\rho) = \mathcal{Q}_k(a,b,\gamma,\rho)$ where the class $\mathcal{Q}_k(a,b,\gamma,\rho)$ is defined by Noor (1999).
(2) $\mathcal{Q}^{0,\mu}_k(\mu+1,1,1,0) = \mathcal{V}_k$ where the class $\mathcal{V}_k$ is

defined by Loewner (1917) which is the class of bounded boundary rotation, and studied by Paatero (Paatero,1931;1932).
(3) $\mathcal{Q}^{0,\mu}_2(\mu+1,1,0,\rho) = S^*(\rho)$.
(4) $\mathcal{Q}^{0,\mu}_2(\mu+1,1,0,\rho) = K(\rho)$.

In the present paper, by using the same techniques of previous work, we extend the results of Lucyna (2013) which are also an extension of the work done by Noor (1999).

II. MAIN RESULTS

The following lemmas will be used in our investigation.

Lemma 1 (Ruscheweyh & Sheil-Small, 1973) *Let $f \in K$, and $g \in S^*$, then for each analytic function H,*
$$\frac{(f*Hg)(\mathbb{U})}{(f*g)(\mathbb{U})} \subset \overline{co}H(\mathbb{U}),$$
where $\overline{co}H(\mathbb{U})$ denotes the closed convex hull of $H(\mathbb{U})$.

Lemma 2 (Ruscheweyh 1982) *Let $0 < a \leq b$. If $b \geq 2$ or $a+b \geq 3$, then the function $\phi(a,b;z)$ belongs to the class K of convex functions.*

Lemma 3 (Noor 1999) Let $f \in \mathcal{P}_k(\alpha), g \in \mathcal{P}_k(\beta)$, *for* $\alpha \leq 1, \beta \leq 1$. *Then* $(f*g) \in \mathcal{P}_k(\delta)$, *where* $\delta = 1-2(1-\alpha)(1-\beta)$.

Theorem 4 *Let* $0 < b_1 \leq b_2, 0 < a_1 \leq a_2, 0 \leq \rho < 1$, *we have*:
(i) If $b_2 \geq 2$ or $b_1+b_2 \geq 3$, *then* $\mathcal{Q}^{m,\mu}_2(a,b_2,\gamma,\rho) \subset \mathcal{Q}^{m,\mu}_2(a,b_1,\gamma,\rho)$
(ii) If $a_2 \geq 2$ or $a_1+a_2 \geq 3$, *then* $\mathcal{Q}^{m,\mu}_2(a_1,b,\gamma,\rho) \subset \mathcal{Q}^{m,\mu}_2(a_2,b,\gamma,\rho)$.

Proof. (i) Let $f \in \mathcal{Q}^{m,\mu}_2(a,b_2,\gamma,\rho)$ and we set

$$P_i(z) = \frac{z(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_i)f(z))' + \gamma z^2(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_i)f(z))''}{(1-\gamma)(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_i)f(z)) + \gamma z(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_i)f(z))'},$$

i=1,2.

From the definition of $\mathcal{Q}^{m,\mu}_2(a,b_2,\gamma,\rho)$, we have $P_2(z) \prec \tau(z) = \frac{1+(1-2\rho)z}{1-z}$, then from definition of subordination there exists an analytic function ω in $\mathbb{U}$ with $|\omega(z)| < 1$ and $\omega(0) = 0$ such that $P_2(z) = \tau(\omega(z))$. Since we can write $\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z) = \phi(b,a;z) * \mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}f(z)$, then we have

$$P_1(z) =$$
$$\frac{z(\phi(b_1,a;z)*\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}f(z))'+\gamma z^2(\phi(b_1,a;z)*\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}f(z))''}{(1-\gamma)(\phi(b_1,a;z)*\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}f(z))+\gamma z(\phi(b_1,a;z)*\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}f(z))'}$$
$$=\frac{z(\phi(b_2,a;z)*\phi(b_1,b_2;z)*\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}f(z))'+\gamma z^2(\phi(b_2,a;z)*\phi(b_1,b_2;z)*\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}f(z))''}{(1-\gamma)(\phi(b_2,a;z)*\phi(b_1,b_2;z)*\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}f(z))+\gamma z(\phi(b_2,a;z)*\phi(b_1,b_2;z)*\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}f(z))'}$$
$$=\frac{z(\phi(b_1,b_2;z)*\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_2)f(z))'+\gamma z^2(\phi(b_1,b_2;z)*\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_2)f(z))''}{(1-\gamma)(\phi(b_1,b_2;z)*\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_2)f(z))+\gamma z(\phi(b_1,b_2;z)*\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_2)f(z))'}$$
$$=\frac{\phi(b_1,b_2;z)*\left[(z\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_2)f(z))'+\gamma z^2(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_2)f(z))''\right]}{\phi(b_1,b_2;z)*\left[(1-\gamma)(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_2)f(z))+\gamma z(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_2)f(z))'\right]}$$

$$\frac{\phi(b_1,b_2;z)*[P_2(z).h(z)]}{\phi(b_1,b_2;z)*h(z)}=\frac{\phi(b_1,b_2;z)*[\tau(\omega(z)).h(z)]}{\phi(b_1,b_2;z)*h(z)}$$

where

$$h(z) =$$
$$(1-\gamma)(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_2)f(z))+\gamma z(\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b_2)f(z))'.$$

It follows from Lemma 2 that the function $\phi(b_1,b_2;z)$ is convex, and it follows from the definition of $\mathcal{Q}_2^{m,\mu}(a,b,\gamma,\rho)$ that $\Re e\{\frac{zh'(z)}{h(z)}\}=\Re e\{P_2(z)\}>\rho\geq 0$ that is $h(z)$ is starlike function of order ρ.

Therefore, applying Lemma 1 we get $\dfrac{\phi(b_1,b_2;z)*[\tau(\omega(z)).h(z)]}{\phi(b_1,b_2;z)*h(z)}\subset\overline{co}\tau(\omega(\mathbb{U}))\subset\tau(\mathbb{U})$, since τ is convex univalent, thus $P_1\prec\tau$, or equivalently, $f\in\mathcal{Q}_2^{m,\mu}(a,b_1,\gamma,\rho)$, which completes the proof of part (i).

(ii) We omit the details of proof for part (ii) since it is similar to part(i). Theorem 4 is proven.

Theorem 5 *Let* $0\leq\rho<1, 0<\mu_1\leq\mu_2$, *and* $\mu_1,\mu_2>-1$. *If* $\mu_2\geq 1$ or $\mu_1+\mu_2\geq 1$, *then* $\mathcal{Q}_2^{m,\mu_2}(a,b,\gamma,\rho)\subset\mathcal{Q}_2^{m,\mu_1}(a,b,\gamma,\rho)$.

Proof. Let $f\in\mathcal{Q}_2^{m,\mu_2}(a,b,\gamma,\rho)$ and we set

$$Q_i(z) =$$
$$\frac{z(\mathcal{I}^{m,\mu_i}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'+\gamma z^2(\mathcal{I}^{m,\mu_i}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))''}{(1-\gamma)(\mathcal{I}^{m,\mu_i}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))+\gamma z(\mathcal{I}^{m,\mu_i}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'},$$
$$i=1,2.$$

From the definition of $\mathcal{Q}_2^{m,\mu_2}(a,b,\gamma,\rho)$, we have $Q_2(z)=\tau(\omega(z))$. Setting $\mathcal{I}^{m,\mu_1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z)=\varphi^{\mu_1}_{\mu_2}(z)*\mathcal{I}^{m,\mu_2}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z)$, where $\varphi^{\mu_1}_{\mu_2}(z)=\sum_{k=1}^{\infty}\frac{(\mu_1+1)_{k-1}}{(\mu_2+1)_{k-1}}z^k$, then we have

$$Q_1(z) =$$
$$\frac{z(\varphi^{\mu_1}_{\mu_2}(z)*\mathcal{I}^{m,\mu_2}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'+\gamma z^2(\varphi^{\mu_1}_{\mu_2}(z)*\mathcal{I}^{m,\mu_2}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))''}{(1-\gamma)(\varphi^{\mu_1}_{\mu_2}(z)*\mathcal{I}^{m,\mu_2}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))+\gamma z(\varphi^{\mu_1}_{\mu_2}(z)*\mathcal{I}^{m,\mu_2}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'}$$

$$=\frac{\varphi^{\mu_1}_{\mu_2}(z)*[z(\mathcal{I}^{m,\mu_2}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'+\gamma z^2(\mathcal{I}^{m,\mu_2}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'']}{\varphi^{\mu_1}_{\mu_2}(z)*[(1-\gamma)(\mathcal{I}^{m,\mu_2}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))+\gamma z(\mathcal{I}^{m,\mu_2}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))']}$$

$$\frac{\varphi^{\mu_1}_{\mu_2}(z)*[Q_2(z).h(z)]}{\varphi^{\mu_1}_{\mu_2}(z)*h(z)}=\frac{\varphi^{\mu_1}_{\mu_2}(z)*[\tau(\omega(z)).h(z)]}{\varphi^{\mu_1}_{\mu_2}(z)*h(z)}.$$

Here

$$h(z) = (1-\gamma)(\mathcal{I}^{m,\mu_2}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z)) +$$

$$\gamma z(\mathcal{I}^{m,\mu_2}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'.$$

It follows from Lemma 2 that the function $\varphi^{\mu_1}_{\mu_2}(z)$ is convex, and it follows from the definition of $\mathcal{Q}^{m,\mu}_2(a,b,\gamma,\rho)$ that

$\Re e\{\frac{zh'(z)}{h(z)}\} = \Re e\{Q_2(z)\} > \rho \geq 0,$

that is $h(z)$ is starlike function of order ρ. Therefore, applying Lemma II.1, we get

$$\frac{\varphi^{\mu_1}_{\mu_2}(z) * [\tau(\omega(z)).h(z)]}{\varphi^{\mu_1}_{\mu_2}(z) * h(z)} \subset \overline{co}\tau(\omega(\mathbb{U})) \subset \tau(\mathbb{U}).$$

Since τ is convex univalent, thus $Q_1 \prec \tau$, or equivalently, $f \in \mathcal{Q}^{m,\mu_1}_2(a,b,\gamma,\rho)$. This completes the proof of Theorem 5.

Theorem 6. *For* $0 \leq \rho < 1, \mu > -1$, *then* $\mathcal{Q}^{m,\mu+1}_2(a,b,\gamma,\rho) \subset \mathcal{Q}^{m,\mu}_2(a,b,\gamma,\rho)$.

Proof. Let $f \in \mathcal{Q}^{m,\mu+1}_2(a,b,\gamma,\rho)$ and we set

$$F_i(z) = \frac{z(\mathcal{I}^{m,\mu+i}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))' + \gamma z^2(\mathcal{I}^{m,\mu+i}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))''}{(1-\gamma)(\mathcal{I}^{m,\mu+i}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z)) + \gamma z(\mathcal{I}^{m,\mu+i}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'},$$

i=1,2.

From the definition of $\mathcal{Q}^{m,\mu+1}_2(a,b,\gamma,\rho)$, we have $F_1(z) = \tau(\omega(z))$. Setting $\mathcal{I}^{m,\mu}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z) = \varphi_\mu(z) * \mathcal{I}^{m,\mu+1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z),$ where $\varphi_\mu(z) = \sum_{k=1}^{\infty} \frac{(\mu+1)_{k-1}}{(\mu+2)_{k-1}} z^k$, then we have

$$F_0(z) = \frac{z(\varphi_\mu(z) * \mathcal{I}^{m,\mu+1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))' + \gamma z^2(\varphi_\mu(z) * \mathcal{I}^{m,\mu+1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))''}{(1-\gamma)(\varphi_\mu(z) * \mathcal{I}^{m,\mu+1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z)) + \gamma z(\varphi_\mu(z) * \mathcal{I}^{m,\mu+1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'}$$

$$= \frac{\varphi_\mu(z) * [z(\mathcal{I}^{m,\mu+1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))' + \gamma z^2(\mathcal{I}^{m,\mu+1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'']}{\varphi_\mu(z) * [(1-\gamma)(\mathcal{I}^{m,\mu+1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z)) + \gamma z(\mathcal{I}^{m,\mu+1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))']}$$

that is

$$\frac{\varphi_\mu(z) * [F_1(z).h(z)]}{\varphi_\mu(z) * h(z)} = \frac{\varphi_\mu(z) * [\tau(\omega(z)).h(z)]}{\varphi_\mu(z) * h(z)}.$$

Here

$h(z) = (1-\gamma)(\mathcal{I}^{m,\mu+1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))$

$+\gamma z(\mathcal{I}^{m,\mu+1}_{\lambda_1,\lambda_2,\ell,d}(a,b)f(z))'.$

It follows from Lemma 2 that the function $\varphi_\mu(z)$ is convex, and it follows from the definition of $\mathcal{Q}^{m,\mu}_2(a,b,\gamma,\rho)$

that $\Re e\{\frac{zh'(z)}{h(z)}\} = \Re e\{F_1(z)\} > \rho \geq 0$ that is $h(z)$ is starlike function of order ρ. Therefore, applying Lemma 1 we get

$$\frac{\varphi_\mu(z) * [\tau(\omega(z)).h(z)]}{\varphi_\mu(z) * h(z)} \subset \overline{co}\tau(\omega(\mathbb{U})) \subset \tau(\mathbb{U}).$$

Since τ is convex univalent, thus $F_0 \prec \tau$, or equivalently, $f \in \mathcal{Q}^{m,\mu}_2(a,b,\gamma,\rho)$. This completes the proof of Theorem 6.

Remark 7 If we put $m = \ell = d = a = b = 1, \lambda_2 = 0,$ in Theorem 5 and Theorem 6, we notice that Al-Abbadi and Darus in (Al-Abbadi & Darus, 2010) obtained the same result, for $h = \frac{1+(1-2\rho)z}{1-z}$, $P^{n_2}_\lambda(h,\beta) \subset P^{n_1}_\lambda(h,\beta)$ and $P^{n+1}_\lambda(h,\beta) \subset P^{n}_\lambda(h,\beta)$.

Theorem 8 Let $a, b > 0, \leq \gamma \leq 1, 0 \leq \rho < 1$, then we have

(1) Let $0 < b_1 \leq b_2$. If $b_2 \geq 2$ or $b_1 + b_2 \geq 3$, then we have

$\mathcal{T}^{m,\mu}_k(a,b_2,\gamma,\rho) \subset \mathcal{T}^{m,\mu}_k(a,b_1,\gamma,\rho).$

(2) Let $0 < a_1 \leq a_2$. *If* $a_2 \geq 2$ *or* $a_1 + a_2 \geq 3$, then we have

$\mathcal{T}^{m,\mu}_k(a_1,b,\gamma,\rho) \subset \mathcal{T}^{m,\mu}_k(a_2,b,\gamma,\rho).$

Proof. Let $f \in \mathcal{T}_k^{m,\mu}(a, b_2, \gamma, \rho)$, and let us define

$$Q_i = (\mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}(a, b_i)f(z))' + \gamma z(\mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}(a, b_i)f(z))'',$$

then from the definition of $\mathcal{T}_k^{m,\mu}(a, b_2, \gamma, \rho)$ we have $Q_2 \in \mathcal{P}_k(\rho)$, or equivalently

$$Q_2 = \left(\frac{k}{4} + \frac{1}{2}\right)\tau_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)\tau_2(z),$$

where $\tau_i \in \mathcal{P}(\rho), i = 1, 2\ldots$. We note that

$$Q_1(z) = (\phi(b_1, a; z) * \mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}f(z))'$$
$$+\gamma z(\phi(b_1, a; z) * \mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}f(z))''$$
$$= (\phi(b_1, b_2; z) * \phi(b_2, a; z) * \mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}f(z))'$$
$$+\gamma z(\phi(b_1, b_2; z) * \phi(b_2, a; z) * \mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}f(z))''$$
$$= \frac{\phi(b_1, b_2; z)}{z} * Q_2(z)$$
$$= \frac{\phi(b_1, b_2; z)}{z} * \left[\left(\frac{k}{4} + \frac{1}{2}\right)\tau_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)\tau_2(z)\right]$$
$$= \left(\frac{k}{4} + \frac{1}{2}\right)\left(\frac{\phi(b_1, b_2; z)}{z} * \tau_1(z)\right)$$
$$- \left(\frac{k}{4} - \frac{1}{2}\right)\left(\frac{\phi(b_1, b_2; z)}{z} * \tau_2(z)\right).$$

By Lemma 2, we have that $\phi(b_1, b_2; z) \in K$ and by using the established relation $f \in K \Rightarrow f \in S^*(1/2) \Rightarrow \Re e\{\frac{f(z)}{z}\} > 1/2$, and by definition of (ρ) we obtain $\frac{\phi(b_1,b_2;z)}{z} \in \mathcal{P}(1/2)$. Since $\tau_i(z) \in \mathcal{P}(\rho), i = 1, 2$, then from Lemma 3 with $k = 2$ we get $\frac{\phi(b_1,b_2;z)}{z} * \tau_i(z) \in \mathcal{P}(\delta)$, where $\delta = 1 - 2(1 - 1/2)(1 - \rho) = \rho$, what means that $Q_1 \in \mathcal{P}_k(\rho)$, hence $f \in \mathcal{T}_k^{m,\mu}(a, b_1, \gamma, \rho)$. We omit the details of proofs here since it is similar to that of the first part. Thus, the proof of Theorem 8 is complete.

Theorem 9 Let $f \in \mathcal{T}_k^{m,\mu}(a, b, \gamma, \rho)$, $g \in \mathcal{A}$ and $\Re e\{\frac{g(z)}{z}\} > 1/2$ for $z \in \mathbb{U}$. Then $f * g \in \mathcal{T}_k^{m,\mu}(a, b, \gamma, \rho)$.

Proof. First we assume that $f \in \mathcal{T}_k^{m,\mu}(a, b, \gamma, \rho)$, $g \in \mathcal{A}$ and $\Re e\{\frac{g(z)}{z}\} > 1/2$, and we define

$$T_1(z) = [\mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}(a, b)f(z)]'$$
$$+ \gamma z[\mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}(a, b)f(z)]'',\ z \in \mathbb{U}$$

and

$$T_2(z) = [\mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}(a, b)(f * g)(z)]'$$
$$+ \gamma z[\mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}(a, b)(f * g)(z)]'',\ z \in \mathbb{U}.$$

From the definition of $\mathcal{T}_k^{m,\mu}(a, b, \gamma, \rho)$, we get

$$T_1(z) = \left(\frac{k}{4} + \frac{1}{2}\right)\tau_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)\tau_2(z)$$

where $\tau_i \in \mathcal{P}_(\rho), i = 1, 2$. Note that

$$T_2(z) = [\phi(b, a; z) * \mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}(f * g)(z)]'$$
$$+ \gamma z[\phi(b, a; z) * \mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}(f * g)(z)]''$$
$$= [\phi(b, a; z) * \mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}(f * g)(z)]'$$
$$+ \gamma z[\phi(b, a; z) * \mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}(f * g)(z)]''$$
$$= \frac{g(z)}{z} * [(\phi(b, a; z) * \mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}f(z))'$$
$$+ \gamma z(\phi(b, a; z) * \mathcal{I}_{\lambda_1,\lambda_2,\ell,d}^{m,\mu}f(z))'']$$
$$= \frac{g(z)}{z} * T_1(z)$$

since $\frac{g(z)}{z} \in \mathcal{P}(\frac{1}{2})$ and $T_1(z) \in \mathcal{P}(\rho)$. Then putting $\alpha = \frac{1}{2}, \beta = \rho$ in Lemma 3, we conclude that $T_2(z) \in \mathcal{P}(\rho)$. The proof of Theorem 9 is complete.

Remark 10 We remark that the class of functions in $\mathcal{A}$ with $\Re e\{\frac{f(z)}{z}\} > 1/2$ is known to be equal to the closed convex hull of the convex functions $\overline{co}K$ (Hallenbeck & MacGregor, 1984).

Thus in the last theorem we showed that the class $\mathcal{T}_k^{m,\mu}(a, b, \gamma, \rho)$ is invariant under the convolution with functions of $\overline{co}K$.

Now by applying the relation $f \in K \Rightarrow f \in S^*(1/2) \Rightarrow \Re e\{\frac{f(z)}{z}\} > 1/2$, we get the following corollary:

Corollary 11 Let $f \in \mathcal{T}_k^{m,\mu}(a, b, \gamma, \rho)$. Then

(1) $g \in S^*(1/2) \Rightarrow f * g \in \mathcal{T}_k^{m,\mu}(a,b,\gamma,\rho)$,

(2) $g \in K \Rightarrow f * g \in \mathcal{T}_k^{m,\mu}(a,b,\gamma,\rho)$.

Next, we consider results related to Miller & Mocanu (2000) as follows.

Remark 12 Let a and c be the complex numbers with $c \neq 0, -1, -2, \dots$.

We consider the function defined by $M(a,c;z) = {}_1F_1(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}$.

The function M (a, c; z) is called the confluent (or Kummer) hypergeometric function.

If Re (c)> Re(a)>0, M (a, c; z) can be represented as an integral

$$M(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{tz} t^{a-1} (1-t)^{c-a-1} dt.$$

Miller & Mocanu (2000) showed that for $a, c \in \mathbb{R}, c \geq R(a)$,

$$R(a) = \begin{cases} 2(1-a) & if \quad a < \frac{1}{4}, \\ (1-2a)^2 + \frac{5}{4} & if \quad \frac{1}{4} \leq a < \frac{3}{4}, \\ 2a & if \quad \frac{3}{4} \leq a, \end{cases} \qquad (13)$$

and the function zM(a, c; z) is starlike of order 1/2 in $\mathbb{U}$.

Thus, we immediate obtain the following:

Corollary 13 If $a, c \in R$, and $c \geq R(a)$, where $R(a)$ is given by (13), then $f \in \mathcal{T}_k^{m,\mu}(a,b,\gamma,\rho) \Rightarrow (zM(a,c;z) * f(z)) \in \mathcal{T}_k^{m,\mu}(a,b,\gamma,\rho)$.

Corollary 14 *Let f be given by $f(z) = \sum_{k=2}^{\infty} a_k z^k$ and for $n \in \mathbb{N} \setminus \{1\}$. Let* $\sigma_n(z) = z + \sum_{k=2}^{n} a_k z^k$. If $f \in \mathcal{T}_k^{m,\mu}(a,b,\gamma,\rho)$, then $\frac{\sigma_n(r_n z)}{r_n} \in \mathcal{T}_k^{m,\mu}(a,b,\gamma,\rho)$, where

$$r_n = sup\left\{ r : \Re e\left(\sum_{k=0}^{n-1} z^k \right) > \frac{1}{2}, |z| < 1 \right\}. \qquad (14)$$

Proof. Let $f \in \mathcal{T}_k^{m,\mu}(a,b,\gamma,\rho)$. Putting $g_n(z) = \sum_{k=1}^{n} z^k$, we can write $\sigma_n(z) = (f * g_n)(z)$.

Thus from (14) we get $\Re e\left(\frac{g_n(r_n z)}{r_n z}\right) > \frac{1}{2}$, for $z \in \mathbb{U}, n \in \mathbb{N} \setminus \{1\}$. Then by applying Theorem 9, we get

$$\frac{\sigma_n(r_n z)}{r_n}$$
$$= \frac{1}{r_n}(f * g_n)(r_n z)$$
$$= f(z) * \frac{g_n(r_n z)}{r_n} \in \mathcal{T}_k^{m,\mu}(a,b,\gamma,\rho).$$

III. CONCLUSION

In this work we discussed on properties of inclusion and convolution for classes of analytic functions defined by integral operator through the method of Noor (1999). It is based on subordination method of the inverse functions. Many other interesting results can be achieved by introducing other subclasses.

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