ASM Sc. J., 13, 2020
https://doi.org/10.32802/asmscj.2020.sm26(4.26)

Two-Point Block Method for Van der Pol Equation

Nooraini Zainuddin[1*] and Zarina Bibi Ibrahim[2]

[1]*Department of Fundamental and Applied Sciences, Faculty of Science and Information Technology, Universiti Teknologi PETRONAS, 32610 Seri Iskandar, Perak, Malaysia.*

[2]*Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia.*

The second order ordinary differential equations (ODEs) of Van der Pol equation is treated by using the two-point block backward differentiation formula. Two types of Van der Pol equation which are stiff and nonstiff are considered. The main motivation of this study is to solve the Van der Pol equation directly instead of reducing it to a system of first order equation. The two-point block method is implemented in constant step size and will produce two approximated solutions for each step. Some numerical results are presented, and the comparisons are made with the existing solvers for both stiff and nonstiff ODE to validate the numerical performance of the two-point block method.

Keywords: block; direct method; nonstiff; stiff; Van der Pol.

I. INTRODUCTION

Van der Pol (VDP) equation is first introduced in 1920's by the physicist Balthasar van der Pol to describe a simple self-oscillating triode circuit. Since then, this equation has been used to model a variety of physical and biological phenomena such as in the development of the model of the interaction of two plates in a geological fault (Cartwright et al., 1999).

The VDP equation is a nonlinear second order ordinary differential equation (ODE) given as

$$y'' = -y + \mu(1 - y^2)y' \qquad (1)$$

whereμindicates the stiffness of the VDP equation. Occasionally, equation (1) is solved by reducing it into the form of first order system as

$$y_1' = y_2, \quad y_2' = -y_1 + \mu(1 - {y_1}^2)y_2. \qquad (2)$$

Many studies were designed on solving various form of VDP equation. Ramana & Prasad (2014) proposed the modified version of Adomian Decomposition Method to solve the forced and unforced VDP equation with$\mu = 1$. In a study done by He et al. (2016), the variable order fractional VDP is treated by using the method of Adams Bashforth Moulton with $\mu = 2.5$. Mishra et al. (2016) successfully used the Homotopy analysis method to deal with the fractional order VDP equation.

A paper written by Cartwright (1999), noted that for VDP equation, at the large μ, the equation is very stiff and exhibiting a relaxation oscillator. The concept of stiff equations was first introduced by Curtiss & Hirschfelder (1952) where they claim that "stiff equations are equations where certain implicit methods, in particular Backward Differentiation Formulas, perform better, usually tremendously better than explicit ones". Cash (2003) gives the idea of stiff equation as a problem with some smooth and transient solutions. There are numerous works on stiff problems were done since the introduction of stiff equation in 1952. For instances, Ibrahim *et. al.,* (2007) proposed 2-point and 3-point block method for stiff first order ODE and Coudière et al. (2018) proposed Exponential Adams-Bashforth for the stiff ODE system in the models of cardiac electrophysiology.

In this paper, the authors propose to solve the nonstiff and stiff VDP directly by varying the value of μ. The proposed method is Fifth Order Direct Block Backward

*Corresponding author's e-mail: aini_zainuddin@utp.edu.my

Differentiation Formula (5-DBBDF) as proposed by Zainuddin et al. (2016). The numerical results are compared with the existing MATLAB solvers for ODE, which are ode45 for nonstiff ODE and ode15s for the stiff ODE.

II. Fifth Order Direct Block Backward Differentiation Formula (5-DBBDF) Method: A Review

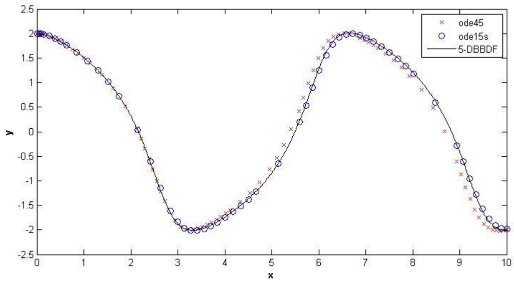
The Fifth Order Direct Block Backward Differentiation Formula (5-DBBDF) is a method specifically design to solve the problem of second order ODE. The method is derived by utilizing up to five back values at the points $x_{n-4}, x_{n-3}, x_{n-2}, x_{n-1}$and x_n. The 5-DBBDF is given as:

$$
\begin{aligned}
& hy'_{n+1} - \frac{1}{6}y_{n+2} - \frac{77}{60}y_{n+1} = -\frac{5}{2}y_n \\
& + \frac{5}{3}y_{n-1} - \frac{5}{6}y_{n-2} + \frac{1}{4}y_{n-3} - \frac{1}{30}y_{n-4}, \\
& y_{n+1} - \frac{137}{147}y_{n+2} + \frac{60}{49}h^2 f_{n+1} = -\frac{85}{49}y_n \\
& + \frac{470}{147}y_{n-1} - \frac{95}{49}y_{n-2} + \frac{31}{49}y_{n-3} - \frac{13}{147}y_{n-4}, \\
& hy'_{n+2} - \frac{49}{20}y_{n+2} + 6y_{n+1} = \frac{15}{2}y_n \\
& - \frac{20}{3}y_{n-1} + \frac{15}{4}y_{n-2} - \frac{6}{5}y_{n-3} + \frac{1}{6}y_{n-4}, \\
& y_{n+2} - \frac{27}{7}y_{n+1} - \frac{45}{203}h^2 f_{n+2} = -\frac{5265}{812}y_n \\
& + \frac{1270}{203}y_{n-1} - \frac{1485}{406}y_{n-2} + \frac{243}{203}y_{n-3} - \frac{137}{812}y_{n-4}.
\end{aligned} \tag{3}
$$

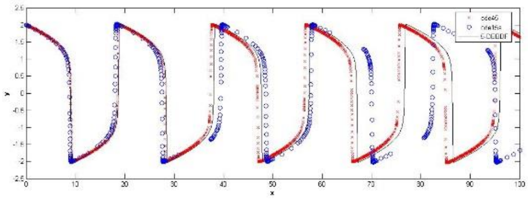
This method solves the second order problem in block term, for which solutions are given at two points simultaneously. The authors had proven that the method is capable to solve the problem of stiff ode as indicated by the numerical result section. However, no numerical result on the nonlinear ODE is given to prove the availability of the method on dealing with nonlinear stiff ODEs.

III. NUMERICAL RESULT

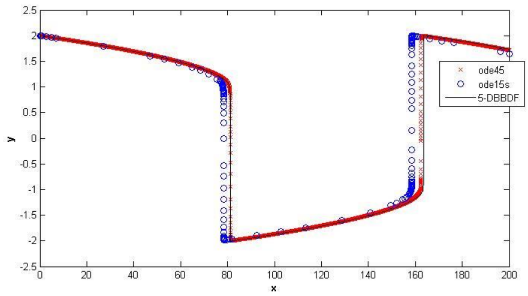
Numerical performance of the 5-DBBDF method on the various values of μ are presented in this section. The values of μ are varied as$\mu = 1, 10, 100, 200$ with the initial conditions$y(0) = 2$and $y'(0) = 0$. The ode45 and ode15s are variable steps solvers while the 5-DBBDF is fixed step method. For figures 1 – 3 below, the numerical plotting is made by using the tolerance (tol) and step size of 0.01 for the $\mu = 1, 10$. For$\mu = 100, 200$, the tolerance used is 0.01, while for 5-DBBDF, the step size used is 0.001.



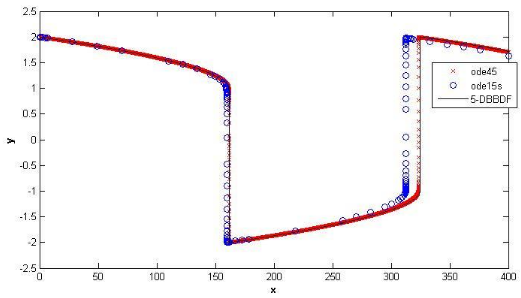
(a)$\mu = 1$



(b)$\mu = 10$

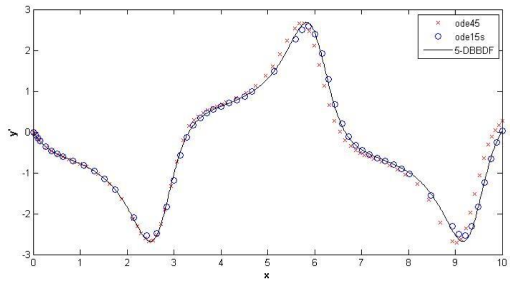


(c)$\mu = 100$

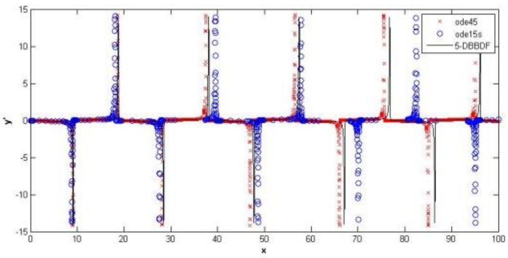


(d)$\mu = 200$

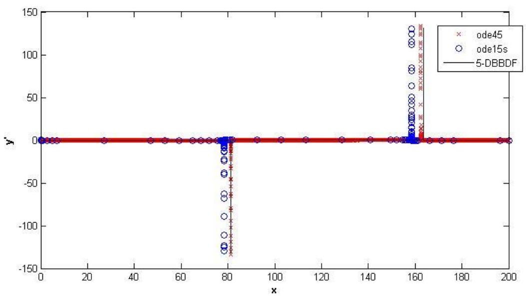
Figure 1. Graph of solution for x and y by using tolerance 0.01 for ode45 and ode15s and step size 0.01 for 5-DBBDF for $\mu = 1, \mu = 10$, and step size 0.001 for 5-DBBDF for $\mu = 100, \mu = 200$



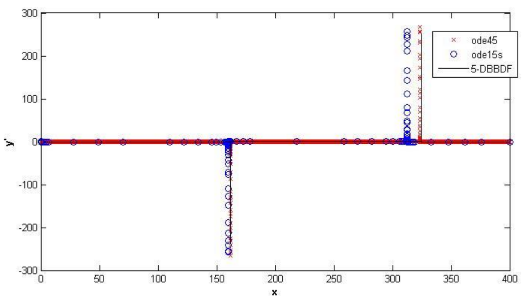
(a)$\mu = 1$



(b) $\mu = 10$

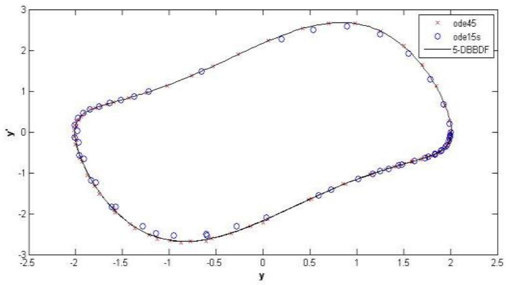


(c)$\mu = 100$

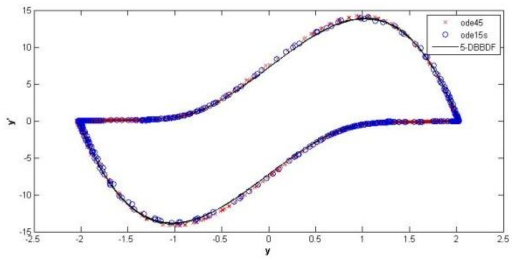


(d)$\mu = 200$

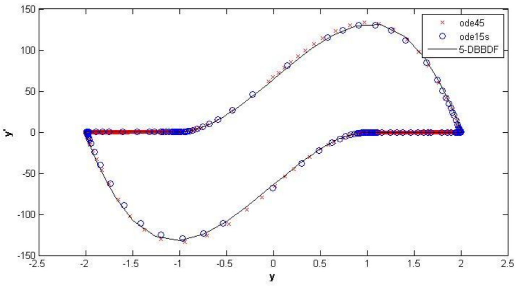
Figure2. Graph of solution for x and y' by using tolerance 0.01 for ode45 and ode15s and step size 0.01 for 5-DBBDF for $\mu = 1, \mu = 10$, and step size 0.001 for 5-DBBDF for $\mu = 100, \mu = 200$.



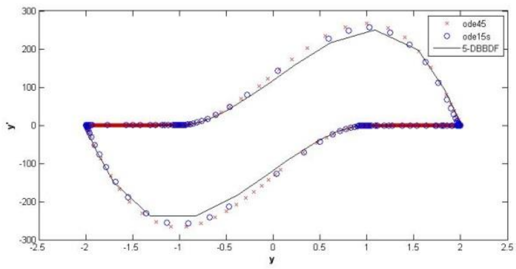
(a)$\mu = 1$



(b)$\mu = 10$



(c)$\mu = 100$



(d)$\mu = 200$

Figure3. Graph of solution for y and y' by using tolerance 0.01 for ode45 and ode15s and step size 0.01 for 5-DBBDF for $\mu = 1, \mu = 10$, and step size 0.001 for 5-DBBDF for $\mu = 100, \mu = 200$.

Table 1. Values of y and y' at end of interval for $\mu = 1$.

Tol/ Step Size	Method	y	y'
0.01	ode45	-1.99588	2.64076E-01
	ode15s	-1.97900	1.88920E-02
	5-DBBDF	**-2.00**905	**3**.00933E-02
0.001	ode45	**-2.01**299	2.06769E-02
	ode15s	**-2.0086**4	2.44333E-02
	5-DBBDF	**-2.00835**	**3.2**8777E-02
0.0001	ode45	**-2.00833**	**3**.32591E-02
	ode15s	**-2.00835**	**3.2**4814E-02
	5-DBBDF	**-2.0083**4	**3.29**068E-02
0.00001	ode45	**-2.00834**	**3.29**342E-02
	ode15s	**-2.0083**1	**3.29**217E-02
	5-DBBDF	**-2.0083**4	**3.29**093E-02

Table 2. Values of y and y' at end of interval for $\mu = 10$.

Tol/ Step Size	Method	y	y'
0.01	ode45	**1.63**861	**-9**.43305E-02
	ode15s	-1.68083	9.03342E-02
	5-DBBDF	1.75409	-8.40846E-02
0.001	ode45	**1.64**412	**-9**.57840E-02
	ode15s	1.57360	-1.05176E-01
	5-DBBDF	**1.64**312	**-9**.59645E-02
0.0001	ode45	**1.64**165	**-9.61**515E-02
	ode15s	**1.62**665	-9.80003E-02
	5-DBBDF	**1.64**092	**-9.62**381E-02
0.00001	ode45	**1.64**089	**-9.62**419E-02
	ode15s	**1.63**874	**-9.65**102E-02
	5-DBBDF	**1.64**089	**-9.62**857E-02

Table 3. Values of y and y' at end of interval for $\mu = 100$.

Tol/ Step Size	Method	y	y'
0.01	ode45	**1.71**715	-4.47473E-04
	ode15s	1.64776	-9.58070E-03
	5-DBBDF	NC	NC
0.001	ode45	**1.72**087	**-8.7**2080E-03
	ode15s	**1.70**908	**-8.8**8933E-03
	5-DBBDF	**1.7**3559	**-8**.62459E-03
0.0001	ode45	**1.718**75	**-8.7**5417E-03
	ode15s	**1.718**25	**-8.79**914E-03
	5-DBBDF	**1.718**91	**-8.79**366E-03
0.00001	ode45	**1.718**59	**-8.79**476E-03
	ode15s	**1.718**34	**-8.79**890E-03
	5-DBBDF	**1.718**59	**-8.79**679E-03

*NC=Not converge

Table 4. Values of y and y' at end of interval for $\mu = 200$.

Tol/ Step Size	Method	y	y'
0.01	ode45	**1.7**0501	-3.37740E-03
	ode15s	**1.6**3208	**-4**.84997E-03
	5-DBBDF	NC	NC
0.001	ode45	**1.710**69	**-4.4**0335E-03
	ode15s	1.69660	**-4**.50429E-03
	5-DBBDF	**1.7**2117	**-4**.38528E-03
0.0001	ode45	**1.710**93	**-4.4**2442E-03
	ode15s	**1.70**773	**-4.4**5501E-03
	5-DBBDF	**1.71**201	**-4.43**323E-03
0.00001	ode45	**1.710**79	**-4.43**850E-03
	ode15s	**1.710**83	**-4.43**886E-03
	5-DBBDF	**1.710**80	**-4.43**933E-03

*NC=Not converge

IV. DISCUSSION

All three methods manage to solve the VDP equation given $\mu = 1, 10, 100, 200$. The ode45 is specially designed for nonstiff ODE and ode15s is specifically for stiff ODE. From figures 1-3, at the values of $\mu = 100{,}200$, the steps taken for ode45 are so small that the individual crosses representing each step merge to become a continuous broad line. However, the ode15s only take small steps at the changing phase of fast and slow states. This shows that for $\mu = 1, 10$, the VDP equation is nonstiff, while for $\mu = 100, 200$,the VDP equation is stiff.

From figures 1 – 3, the numerical performance of the 5-DBBDF had shown its potential on solving stiff and nonstiff nonlinear second order ODE directly. The graph of the numerical solutions agrees with the results obtained with the ode45 and ode15s. When $\mu = 1$ and tolerance/step size used is 0.01, the graph of 5-DBBDF follow closely the graph of ode45 and ode15s. Referring to tables 1, 2, 3 and 4, by comparing the numerical solutions at the end of interval, the 5-DBBDF manages to get the approximation values that equivalent to the ode45 and ode15s up until four decimal values at lower tolerance/step size for y and up to two decimal values for y'.

The graphs when $\mu = 10$ and the interval up to $x = 100$are plotted for the tolerance and step size of 0.01. The oscillation of the numerical is clearly shown. However, it is clearly seen that the result from ode15s deviates from the result of ode45 and 5-DBBDF. The numerical values from tables 1 and 2 conclude that the result should follow the result of ode45 and 5-DBBDF.As the tolerance/step size becomes smaller, the result agrees with the values given by the ode45 and 5-DBBDF.

The graphs when $\mu = 100, 200$, are given with the interval up to $x = 200, 400$ respectively. The 5-DBBDF do not converge for the step size 0.01. However, graphs and tables of the numerical solution indicate that even the 5-DBBDF requires step size 0.001 to give the approximated values, the solutions agreed to the same decimal values as given by ode45 and ode15s.

VI. CONCLUSION

Numerical performance given by the 5-DBBDF had proven its availability on solving nonstiff and stiff nonlinear VDP equation. Although the 5-DBBDF requires smaller step size to converge at higher μ, the approximated solutions still follow closely the results given by ode45 and ode15s. Since the 5-DBBDF is fixed step method, it is recommended that the implementation is further extended to variable step so that it will able to give numerical solution at higher tolerance and at the same time reducing the computational cost.

V. REFERENCES

Cartwright, J. 1999, Nonlinear stiffness, Lyapunov exponents, and attractor dimension. Physics Letters A, 264, 298 – 302.

Cartwright, J., Eguiluz, V., Hernandez-Garcia, E. & Piro, O. 1999, Dynamics of elastic excitable Media. J. Bifur. Chaos Appl. Sci. Engrg., 9, 2197 - 2202.

Cash, J. R. 2003, Review Paper: Efficient numerical methods for the solution of stiff initial-value problems and differential algebraic equations. Proc. R. Soc. Lond. A, 459, 797 – 815.

Coudière, Y., Douanla-Lontsi, C., & Pierre, C. 2018, Exponential Adams-Bashforth integrators for stiff ODEs, application to cardiac electrophysiology. Mathematics and Computers in Simulation, 153, 15 – 34.

Curtiss, C. F. & Hirschfelder, J. O. 1952, Integration of Stiff Equations. National Academy of Sciences, 38, 235 – 243.

He, L., Yi, L., & Tang, P. 2016, Numerical scheme and dynamic analysis for variable-order fractional van der pol model of nonlinear economic cycle. Advances in Differences Equations, 195, 1 – 11.

Ibrahim, Z. B., Suleiman, M. B., & Othman, K. I. 2007, Implicit r –point block backward differentiation formula for solving first order stiff ordinary differential equations. Applied Mathematics and Computation, 186, 558 – 565.

Mishra, V., Das, S., Jafari, H., & Ong, S. H. 2016, Study of fractional order van der pol equation. Journal of King Saud University – Science, 28, 55 – 60.

Ramana, P. V. & Prasad, B. K. 2014, Modified Adomian decomposition method for van der pol equations. International Journal of Non-Linear Mechanics, 65, 121 – 132.

Zainuddin, N., Ibrahim, Z. B., Othman, K. I., & Suleiman, M. 2016, Direct fifth order block backward differentiation formulas for solving second order ordinary differential equations. Chiang Mai J. Sci., 43, 1171 – 1181.