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Development of Novel Subclasses for Bi-Univalent Functions

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This manuscript presents the development of new subclasses for bi-univalent functions and the subclasses are closely related to Chebyshev polynomials having Al-Oboudi differential operator. The functions contained in the subclasses were used to account for the initial coefficient estimates of $|a_2|$and $|a_3|$.

Keywords: coefficient estimates; subordination; Chebyshev Polynomials; bi-univalent; Al-Oboudi Operator

I. INTRODUCTION

Let $\mathbb{C}$ be a set of complex numbers, $\mathbb{R} = (-\infty, \infty)$, a set of real numbers and $\mathbb{N} := \{1,2,3,\ldots\} = \mathbb{N}_0 \backslash \{0\}$ representing a set of positive integers. Then, let A be defined as $\Delta = \{z : z \in \mathbb{C}, |z| < 1\}$, and an open unit disc denoted as a class function expressed in (1).

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1)$$

The subclass of A is represented by S, which is a normalised analytic function as shown in $f'(0) = 1$ and $f(0) = 0$. Given that $K(\alpha)$ and $S^*(\alpha)$ are expressed as the convex and starlike functions respectively in the order of $\alpha(0 \leq \alpha < 1)$, they can be represented by the renowned subclasses of S.

For $(f \in A)$, Al-Oboudi (2004) represented the operator as thus:

$$D_\delta^0 f(z) = f(z),$$

$$D_\delta^1 f(z) = (1-\delta) f(z) + \delta z f'(z) := D_\delta f(z) (\delta \geq 0) \quad (2)$$

$$D_\delta^n f(z) = D_\delta \left(D_\delta^{n-1} f(z) \right) (n \in \mathbb{N}) \quad (3)$$

As f is represented in (1), it can be seen from (2) and (3) that:

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k \quad (n \in \mathbb{N}_0) \quad (4)$$

where $D_\delta^n f(0) = 0$. The Sălăgean's differential operator is obtained when:

$\delta = 1$. (Sălăgean, 1983).

A function $f : \Delta \to \mathbb{C}$ is called univalent on Δ (or schlict or one-to-one) if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in \Delta$ with $z_1 \neq z_2$. Based on Duren (1983), the theorem of Koebe's one-quarter showed the images of Δ for every univalent function, $(f \in S)$enclosing 1/4 radius disc. Hence, every inverse function of f^{-1} in $(f \in A)$ can be defined as:

$$f^{-1}\big(f(z)\big) = z \quad (z \in U),$$

$$f\big(f^{-1}(w)\big) = w$$

$$\left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

Furthermore, f^{-1} is shown as:

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots \quad (5)$$

A function $(f \in A)$ is called bi-univalent in Δ, if every of f or f^{-1} is univalent. Thus, the notation of bi-univalent functions class is expressed as $\sum$. The background and previous works on $\sum$ can be found in Srivastava *et al.* (2010) and Brannan and Taha (1986; 1988). Indeed, the research

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findings of Srivastava *et al.* (2010) have been used as the basis of revitalisation for the research of numerous subclasses of bi-univalent functions class Σ. Several researchers that conducted similar studies included: Aldawish *et al.* (2020), Khan *et al.* (2020), Hern and Janteng (2020), Omar *et al.* (2019), Porwal and Darus (2013), Bulut (2013), Çağlar *et al.* (2013), Hayami and Owa (2012), Xu *et al.* (2012a), Xu *et al.* (2012b) and Frasin and Aouf (2011).

Next, the concept of subordination was used, given that $f(z) \prec g(z)$, with f being a subordinate to g, and both functions taken to be analytic. This implies that $f(z) = g(w(z))$, where w is taken as analytic in Δ, which corresponds to $|w(z)| < 1$ and $w(0) = 0$.

Chebyshev polynomials applied in this study, is very relevant in numerical analysis. Chebyshev orthogonal polynomials are associated with important findings on $T_n(x)$ and $U_n(x)$ expressed as shown in (6).

$$T_n(x) = cos\, n\,\theta, \quad U_n(x) = \frac{sin(n+1)\theta}{sin\,\theta} \tag{6}$$

where $(-1 < x < 1)$ is expressed as $x = cos\,\theta$, and subscript n indicated the degree of the polynomial. With the use of the function: $H(z,t) = \frac{1}{1-2tz+z^2}$, we identified that when $t = cos\,\alpha$, $\alpha \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, thus, for all $z \in \Delta$,

$$H(z,t) = 1 + \sum_{n=1}^{\infty} \frac{sin(n+1)\alpha}{sin\,\alpha} z^n = 1 + 2\,cos\,\alpha\, z + (3\,cos^2\alpha - sin^2\alpha)z^2 + \ldots$$

hence, we have:

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots$$
$$\big(z \in \Delta, t \in (-1,1)\big)$$

for $U_{n-1} = \frac{sin(n\, arccos\, t)}{\sqrt{1-t^2}}$, $n \in \mathbb{N}$ and equally

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t), \text{ where:}$$
$$U_1(t) = 2t;$$
$$U_2(t) = 4t^2 - 1,$$
$$U_3(t) = 8t^3 - 4t, \ldots \tag{7}$$

Next, $T_n(t)$ that generated the function is represented as:

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1-tz}{1-2tz+z^2} \quad (z \in \Delta),$$

with $t \in [-1, 1]$. Nevertheless, $T_n(t)$ and $U_n(t)$ of the Chebyshev polynomials have the following relationships:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t),$$
$$T_n(t) = U_n(t) - tU_{n-1}(t),$$
$$2T_n(t) = U_n(t) - U_{n-2}(t).$$

More details on the applications of Chebyshev polynomials can be found in the studies of Doha (1994) and Mason (1967).

Inspired by the recent findings on the bi-univalent functions by Güney *et al.* (2017), Altinkaya and Yalcin (2016), Dziok *et al.* (2015), Murugusundaramoorthy *et al.* (2015), and Vijaya *et al.* (2014), we proposed the new subclasses of $\sum$ and determined the initial coefficients of $|a_2|$ and $|a_3|$ by the application of Chebyshev polynomials associated with Al-Oboudi differential operator.

II. METHODS

The main results of the subclasses were obtained by the application of the following definitions:

Definition 1.
For $0 \le \lambda \le 1$, $\delta \ge 0$, $n \in \mathbb{N}$ and $t \in (-1,1)$, $(f \in \Sigma)$ of the form (1) is said to be in the class of $N_{\Sigma}^{\delta}[n, \lambda, H]$, if:

$$(1-\lambda)\frac{D_\delta^{n+1} f(z)}{D_\delta^n f(z)} + \lambda \frac{D_\delta^{n+2} f(z)}{D_\delta^{n+1} f(z)} \prec H(z,t),$$

$$(1-\lambda)\frac{D_\delta^{n+1} g(w)}{D_\delta^n g(w)} + \lambda \frac{D_\delta^{n+2} g(w)}{D_\delta^{n+1} g(w)} \prec H(w,t) \tag{8}$$

where D_δ^n is the Al-Oboudi operator, g as specified in (5) and $z, w \in \Delta$.

Remark 1.
The new subclasses of $\sum$ are introduced, by specifying the elements of λ and n in Definition 1, taking $t \in (-1, 1)$ and $f(z) \in \Sigma$.

(i) $N_\Sigma^1[n, 0, H] \equiv M_\Sigma^k(0, \Phi(z,t))$ (Güney *et al.*, 2017)

(ii) $N_\Sigma^1[n, 1, H] \equiv M_\Sigma^k\big(1, \Phi(z,t)\big)$ (Güney *et al.*, 2017)

(iii) $N_\Sigma^1[0, \lambda, H] \equiv M_\Sigma^0\big(\lambda, \Phi(z,t)\big)$ (Güney *et al.*, 2017)

(iv) $N_\Sigma^1[0, 0, H] \equiv M_\Sigma^0\big(0, \Phi(z,t)\big)$ (Güney *et al.*, 2017)

(v) $N_\Sigma^1[0,1, H] \equiv M_\Sigma^0\big(1, \Phi(z,t)\big)$ (Güney *et al.*, 2017)

Definition 2.

For $0 \leq \beta \leq 1$ and $t \in (-1, 1)$, a function, $f \in \Sigma$ as contained in (1) is said to be in the class $F_{\Sigma}^{\delta}[n, \beta, H]$, if the following subordinations apply:

$$(1-\beta)\frac{D_{\delta}^{n}f(z)}{z} + \beta\left(D_{\delta}^{n}f(z)\right)' \prec H(z,t),$$

$$(1-\beta)\frac{D_{\delta}^{n}g(w)}{w} + \beta(D_{\delta}^{n}g(w))' \prec H(w,t),$$

D_{δ}^{n} is denoted as the Al-Oboudi operator, g as expressed in (5) and $z, w \in \Delta$.

Remark 2.

Consequently, taking $t \in (-1, 1)$and $f(z) \in \Sigma$, and β and n as expressed in Definition 2, we have the following subclasses of $\sum$ as listed below:

(i) $F_{\Sigma}^{1}[n, 0, H] \equiv F_{\Sigma}^{k}\left(0, \Phi(z,t)\right)$ (Güney *et al.*, 2017)

(ii) $F_{\Sigma}^{1}[n, 1, H] \equiv F_{\Sigma}^{k}\left(1, \Phi(z,t)\right)$ (Güney *et al.*, 2017)

(iii) $F_{\Sigma}^{1}[0, \beta, H] \equiv F_{\Sigma}^{0}\left(\beta, \Phi(z,t)\right)$ (Güney *et al.*, 2017)

(iv) $F_{\Sigma}^{1}[0, 0, H] \equiv F_{\Sigma}^{0}[0, \Phi(z,t)]$ (Güney *et al.*, 2017)

(v) $F_{\Sigma}^{1}[0, 1, H] \equiv F_{\Sigma}^{0}\left(1, \Phi(z,t)\right)$ (Güney *et al.*, 2017)

III. RESULTS

The main results of the subclasses are presented below:

Theorem 1.

Let f expressed in (1) be in the class $N_{\Sigma}^{\delta}[n, \lambda, H]$ and $t \in (0, 1)$. Based on these:

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\begin{bmatrix} 2\delta(1+2\lambda\delta)[1+2\delta]^{n} \\ -\delta\begin{pmatrix}\lambda^2\delta^3 + 3\lambda\delta^2 \\ +2\lambda\delta + \delta + 1\end{pmatrix}[1+\delta]^{2n}\end{bmatrix} 4t^2 \\ +\delta^2(1+\lambda\delta)^2[1+\delta]^{2n}}},$$

and,

$$|a_3| \leq \frac{4t^2}{\delta^2(1+\lambda\delta)^2[1+\delta]^{2n}} + \frac{t}{\delta(1+2\lambda\delta)[1+2\delta]^{n}}.$$

where $0 \leq \lambda \leq 1$.

Proof.

Utilising (8), we have:

$$(1-\lambda)\frac{D_{\delta}^{n+1}f(z)}{D_{\delta}^{n}f(z)} + \lambda\frac{D_{\delta}^{n+2}f(z)}{D_{\delta}^{n+1}f(z)} = H(u(z),t), \quad (9)$$

$$(1-\lambda)\frac{D_{\delta}^{n+1}g(w)}{D_{\delta}^{n}g(w)} + \lambda\frac{D_{\delta}^{n+2}g(w)}{D_{\delta}^{n+1}g(w)} = H(v(w),t). \quad (10)$$

$u(z)$ and $v(w)$ are denoted as:

$$u(z) = c_1 z + c_2 z^2 + \ldots, \quad (11)$$

$$v(w) = d_1 w + d_2 w^2 + \ldots \quad (12)$$

where:

$$|u(z)| < 1,$$

$$|v(w)| < 1,$$

$z \in \Delta$ and $u(0) = v(0) = 0$ are analytic in Δ. It is equally notable that:

$$|u(z)| = |c_1 z + c_2 z^2 + \cdots| < 1,$$

$$|v(w)| = |d_1 w + d_2 w^2 + \ldots| < 1,$$

$$z, w \in \Delta,$$

where

$$|c_j| \leq 1, |d_j| \leq 1 \quad (13)$$

$$\forall j \in \mathbb{N}.$$

Using (11) and (12) in (9) and (10), respectively, we have:

$$(1-\lambda)\frac{D_{\delta}^{n+1}f(z)}{D_{\delta}^{n}f(z)} + \lambda\frac{D_{\delta}^{n+2}f(z)}{D_{\delta}^{n+1}f(z)}$$
$$= 1 + U_1(t)u(z) + U_2(t)u^2(z) + \cdots,$$

$$(14)$$

$$(1-\lambda)\frac{D_{\delta}^{n+1}g(w)}{D_{\delta}^{n}g(w)} + \lambda\frac{D_{\delta}^{n+2}g(w)}{D_{\delta}^{n+1}g(w)}$$
$$= 1 + U_1(t)v(w) + U_2(t)v^2(w) + \cdots$$

In consideration of (1), (4), (5), (7) and (14), we have:

$$1 + \delta(1+\lambda\delta)[1+\delta]^{n}a_2 z$$
$$+ \begin{bmatrix} 2\delta(1+2\lambda\delta)[1+2\delta]^{n}a_3 \\ -\delta(1+2\lambda\delta+\lambda\delta^2)[1+\delta]^{2n}a_2^2\end{bmatrix} z^2 + \cdots$$
$$= 1 + U_1(t)c_1 + [U_1(t)c_2 + U_2(t)c_1^2]z^2 + \cdots,$$
$$1 - \delta(1+\lambda\delta)[1+\delta]^{n}a_2 w$$
$$+ \begin{Bmatrix} \begin{bmatrix} 4\delta(1+2\lambda\delta)[1+2\delta]^{n} \\ -\delta(1+2\lambda\delta+\lambda\delta^2)[1+\delta]^{2n}\end{bmatrix} a_2^2 \\ -2\delta(1+2\lambda\delta)[1+2\delta]^{n}a_3 \end{Bmatrix} w^2 + \cdots$$
$$= 1 + U_1(t)d_1 w + [U_1(t)d_2 + U_2(t)d_1^2]w^2 + \ldots,$$

Thus:

$$\delta(1+\lambda\delta)[1+\delta]^{n}a_2 = U_1(t)c_1, \quad (15)$$

$$2\delta(1+2\lambda\delta)[1+2\delta]^{n}a_3$$
$$-\delta(1+2\lambda\delta+\lambda\delta^2)[1+\delta]^{2n}a_2^2 \quad (16)$$
$$= U_1(t)c_2 + U_2(t)c_1^2$$

$$-\delta(1+\lambda\delta)[1+\delta]^{n}a_2 = U_1(t)d_1, \quad (17)$$

$$\begin{bmatrix}4\delta(1+2\lambda\delta)[1+2\delta]^n \\ -\delta(1+2\lambda\delta+\lambda\delta^2)[1+\delta]^{2n}\end{bmatrix} a_2^2$$

$$-2\delta(1+2\lambda\delta)[1+2\delta]^n a_3 \quad (18)$$

$$= U_1(t)d_2 + U_2(t)d_1^2.$$

By utilising (15) and (17), we have:

$$c_1 = -d_1, \quad (19)$$

$$2[\delta+\lambda\delta^2]^2[1+\delta]^{2n}{a_2}^2 = U_1^2(t)[c_1^2+d_1^2]. \quad (20)$$

By the addition of (16) and (18) and inputting into (20), gave rise to the below expression:

$$a_2^2 = \frac{U_1^3(c_2+d_2)}{2\begin{bmatrix}\begin{Bmatrix}2\delta(1+2\lambda\delta)[1+2\delta]^n \\ -\delta(1+2\lambda\delta+\lambda\delta^2)[1+\delta]^{2n}\end{Bmatrix} U_1^2(t) \\ -\delta^2(1+\lambda\delta)^2[1+\delta]^{2n}U_2(t)\end{bmatrix}}.$$

Taking c_2 and d_2, as expressed in (13) and inputting into (7), resulted to:

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{\left|\begin{matrix}\begin{bmatrix}2\delta(1+2\lambda\delta)[1+2\delta]^n \\ -\delta(\lambda^2\delta^3+3\lambda\delta^2+2\lambda\delta+\delta+1)[1+\delta]^{2n}\end{bmatrix} 4t^2 \\ +\delta^2(1+\lambda\delta)^2[1+\delta]^{2n}\end{matrix}\right|}}.$$

By the subtraction of (18) from (16) and inputting into (19) and (20), resulted to:

$$a_3 = \frac{U_1^2(t)(c_1^2+d_1^2)}{2\delta^2(1+\lambda\delta)^2[1+\delta]^{2n}} + \frac{U_1(t)(c_2-d_2)}{4\delta(1+2\lambda\delta)[1+2\delta]^n}$$

Using (7), and once again applying (13) to the coefficients c_1,c_2,d_1, and d_2, resulted to:

$$|a_3| \le \frac{4t^2}{\delta^2(1+\lambda\delta)^2[1+\delta]^{2n}} + \frac{t}{\delta(1+2\lambda\delta)[1+2\delta]^n}$$

Based on theorem 5, the following corollaries were obtained.

Corollary 1.

Let f be in the class $N_\Sigma^\delta[n,0,H]$, then:

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{\left|\begin{matrix}[2\delta[1+2\delta]^n - \delta(\delta+1)[1+\delta]^{2n}]4t^2 \\ +\delta^2[1+\delta]^{2n}\end{matrix}\right|}},$$

$$|a_3| \le \frac{4t^2}{\delta^2[1+\delta]^{2n}} + \frac{t}{\delta[1+2\delta]^n}.$$

Taken $\delta = 1$, resulted to $M_\Sigma^k(0,\Phi(z,t))$ as introduced in theorem 5 (Güney *et al.*, 2017).

Corollary 2.

Let f be in the class $F_\Sigma^\delta[n,1,H]$, then:

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{\left|\begin{matrix}\begin{bmatrix}2\delta[1+2\delta]^{n+1} \\ -\delta(\delta^3+3\delta^2+3\delta+1)[1+\delta]^{2n}\end{bmatrix} 4t^2 \\ +\delta^2[1+\delta]^{2(n+1)}\end{matrix}\right|}},$$

$$|a_3| \le \frac{4t^2}{\delta^2[1+\delta]^{2(n+1)}} + \frac{t}{\delta[1+2\delta]^{n+1}}$$

Given $\delta = 1$, resulted to $F_\Sigma^k(1,\Phi(z,t))$ as introduced in theorem 5 (Güney *et al.*, 2017).

Corollary 3.

Let f be in the class $N_\Sigma^\delta[0,\lambda,H]$, then:

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{\left|\begin{matrix}\delta^2(1+\lambda\delta)^2 \\ -\delta(\delta^3\lambda^2+3\lambda\delta^2-2\lambda\delta+\delta-1)4t^2\end{matrix}\right|}},$$

$$|a_3| \le \frac{4t^2}{\delta^2(1+\lambda\delta)^2} + \frac{t}{\delta(1+2\lambda\delta)},$$

where $0 \le \lambda \le 1$.

Given $\delta = 1$, resulted to $M_\Sigma^0(\lambda,\Phi(z,t))$ as introduced in theorem 5 (Güney *et al.*, 2017).

Corollary 4.

Taken f to be in the class $N_\Sigma^\delta[0,0,H]$, then:

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{|\delta^2-\delta(\delta-1)4t^2|}},$$

$$|a_3| \le \frac{4t^2}{\delta^2} + \frac{t}{\delta}.$$

Taken $\delta = 1$, then, we had $M_\Sigma^0(0,\Phi(z,t))$ as introduced in theorem 5 (Güney *et al.*, 2017).

Corollary 5.

Taken f to be in the class $N_\Sigma^\delta[0,1,H]$, then:

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{|\delta^2(1+\delta)^2-\delta(\delta^3+3\delta^2-2\delta+\delta-1)4t^2|}},$$

and

$$|a_3| \le \frac{4t^2}{\delta^2(1+\delta)^2} + \frac{t}{\delta(1+2\delta)}.$$

Taken $\delta = 1$, and $t \ne 1/\sqrt{2}$, resulted to $M_\Sigma^0(1,\Phi(z,t))$ as introduced in theorem 5 (Güney *et al.*, 2017).

Theorem 2.

Taken f as contained in (1) to be in the class $F_{\Sigma}^{\delta}[n,\beta,H)]$ and $t\in(0,1)$ resulted to:

$$|a_2|\leq\frac{2t\sqrt{2t}}{\sqrt{\left|\begin{array}{l}[(1+2\beta)[1+2\delta]^n-(1+\beta)^2[1+\delta]^{2n}]4t^2\\+(1+\beta)^2[1+\delta]^{2n}\end{array}\right|}} \quad (21)$$

and

$$|a_3|\leq\frac{4t^2}{(1+\beta)^2[1+\delta]^{2n}}+\frac{2t}{(1+2\beta)[1+2\delta]^n}. \quad (22)$$

Proof.

According to the proofs obtained in theorem (5), we acquired the succeeding expressions below:

$$(1+\beta)[1+\delta]^n a_2=U_1(t)c_1, \quad (23)$$

$$(1+2\beta)[1+2\delta]^n a_3=U_1(t)c_2+U_2(t)c_1^2, \quad (24)$$

$$-(1+\beta)[1+\delta]^n a_2=U_1(t)d_1, \quad (25)$$

$$2(1+2\beta)[1+2\delta]^n a_2^2-(1+2\beta)[1+2\delta]^n a_3= U_1(t)d_2+U_2(t)d_1^2. \quad (26)$$

From (23) and (25), it was established that:

$$c_1=-d_1, \quad (27)$$

$$2(1+\beta)^2[1+\delta]^{2n}a_2^2=U_1^2(t)(c_1^2+d_1^2). \quad (28)$$

Then, the utilisation of (24), (26), and (28) resulted to:

$$a_2^2=\frac{U_1^3(t)(c_2+d_2)}{2\left[\begin{array}{l}(1+2\beta)[1+2\delta]^nU_1^2(t)\\-(1+\beta)^2[1+\delta]^{2n}U_2(t)\end{array}\right]}.$$

Then, by the use of (7) and (13) for the coefficients c_2 and d_2, we obtained the sought-after bound on $|a_2|$ as stated in (21), and by subtracting (26) from (24), and inputting into (27) and (28), resulted to:

$$a_3=\frac{U_1^2(t)(c_1^2+d_1^2)}{2(1+\beta)^2[1+\delta]^{2n}}+\frac{U_1(t)(c_2-d_2)}{2(1+2\beta)[1+2\delta]^n}.$$

Reiteratively, from the use of (7) and (13) for the coefficients c_1, c_2, d_1, and d_2, we got $|a_3|$ as contained in (22).

Corollary 6.

Given f to be in the class $F_{\Sigma}^{\delta}[n,0,H]$, resulted to:

$$|a_2|\leq\frac{2t\sqrt{2t}}{\sqrt{|[[1+2\delta]^n-[1+\delta]^{2n}]4t^2+[1+\delta]^{2n}|}}$$

$$|a_3|\leq\frac{4t^2}{[1+\delta]^{2n}}+\frac{2t}{[1+2\delta]^n}.$$

$F_{\Sigma}^{1}[n,0,H]\equiv F_{\Sigma}^{k}(0,\Phi(z,t))$

Taken $\delta=1$, we had $F_{\Sigma}^{k}(0,\Phi(z,t))$ as introduced in theorem 11 (Güney *et al.,* 2017).

Corollary 7.

Given f to be in the class $F_{\Sigma}^{\delta}[n,1,H]$, then:

$$|a_2|\leq\frac{2t\sqrt{2t}}{\sqrt{|[3[1+2\delta]^n-4[1+\delta]^{2n}]4t^2+4[1+\delta]^{2n}|}}$$

$$|a_3|\leq\frac{t^2}{[1+\delta]^{2n}}+\frac{2t}{3[1+2\delta]^n}.$$

Taken $\delta=1$, resulted to $F_{\Sigma}^{k}(1,\Phi(z,t))$ as introduced in theorem 11 (Güney *et al.,* 2017).

IV. SUMMARY

The bi-univalent functions subclasses of $N_{\Sigma}^{\delta}[n,\lambda,H]$ and $F_{\Sigma}^{\delta}[n,\beta,H)]$ were developed using the subordinations of Chebyshev polynomials, defined by the Al-Oboudi differential operator and by the subclasses applications, the coefficient estimate of $|a_2|$ and $|a_3|$ were determined. The bounds obtained in Theorem 1 and Theorem 2 are the best possible.

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