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On a Subclass of Harmonic Mappings Associated with Hypergeometric Functions

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$HP(\alpha,\beta)$ is a class of functions harmonic and univalent defined in the open unit disc U. Sufficient conditions for a hypergeometric function and an integral operator related to hypergeometric function, to be in the class $HP(\alpha,\beta)$ are derived. Harmonic functions with negative coefficients are also considered in this investigation.

Keywords: Harmonic functions; hypergeometric functions; convolution; integral operator

I. INTRODUCTION

The basic theory of harmonic mappings was initiated in the seminal works of Clunie and Sheil-Small (1984) and Sheil-Small (1990). Since then harmonic univalent functions have been intensively investigated from the point of view of geometric function theory. See for example (Ahuja, 2005; Duren, 2004; Liu & Ponnusamy, 2018; Kayumov & Ponnusamy, 2018; Silverman, 1998) and references therein. In the well-established theory of analytic univalent functions, there are several studies on hypergeometric functions associated with classes of analytic functions (See for example Carlson & Shaffer, 1984; Miller & Mocanu, 1990; Kwon & Cho, 2008; Owa & Srivastava, 1987; Ponnusamy & Ronning, 1998; Ruscheweyh & Singh, 1986; Silverman, 1993; Swaminathan, 2004a, 2004b) investigating univalence, starlikeness and other properties of these functions. On the other hand, only some corresponding studies on connections of hypergeometric functions with harmonic mappings have been done (Ahuja, 2008, 2007; Murugusundaramoorthy & Raina, 2009). Pursuing this line of study, results that bring out connections of hypergeometric functions with a class of harmonic univalent functions considered in (Yalçin & Öztürk, 2004) are established.

Let H be the class of continuous, complex-valued harmonic functions $f(z)=u+iv$ which map the unit disk $U=\{z\in C:|z|<1\}$ onto a domain $D\subset C$. In fact u and v are real harmonic in U. It is well-known (Clunie and Sheil-Small, 1984) that such a harmonic function f can be written as $f=h+\overline{g}$, when h and g are analytic in U. It is also known (Clunie and Sheil-Small, 1984) that a sufficient condition for $f=h+\overline{g}$ to be locally univalent and sense preserving in U is that $|h'(z)|>|g'(z)|$ in U.

Denote by S_H the class of functions $f=h+\overline{g}$ which are harmonic univalent and sense preserving in the unit disk U and f normalized by $f(0)=h(0)=f_z(0)-1=0$. Thus, for $f=h+\overline{g}\in S_H$ we may express the analytic functions

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h and g as

$$h(z)=z+\sum_{n=2}^{\infty}A_n z^n \text{ and } g(z)=\sum_{n=1}^{\infty}B_n z^n \quad (1)$$

where $|B_1|<1$. Note that S_H reduces to the class of normalized analytic univalent functions if the co-analytic part g of f is identically zero. If ϕ_1 and ϕ_2 are analytic and $f=h+\overline{g}$ is in S_H, the convolution or the Hadamard product is defined by

$$f*(\phi_1+\overline{\phi_2})=h*\phi_1+\overline{g*\phi_2}.$$

Let a,b and c be any complex numbers with $c\neq 0,-1,-2,-3,\ldots$. Then the Gauss hypergeometric function written as ${}_2F_1(a,b;c;z)$ or simply as $F_c^{a,b}(z)$ is defined by (i)

$$F_c^{a,b}(z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n, \quad (2)$$

where $(\lambda)_n$ is the Pochhammer symbol given by (ii)

$$(\lambda)_n=\begin{cases}1, & (n=0);\\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & (n=N).\end{cases} \quad (3)$$

Since the hypergeometric series defined in (2) and (3) converges absolutely in U, it follows that $F_c^{a,b}(z)$ defines a function which is analytic in U, provided that c is neither zero nor a negative integer. In fact, $F_c^{a,b}(1)$ converges for $\operatorname{Re}(c-a-b>0)$ and is related to the gamma function by

$$F_c^{a,b}(1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ c\neq 0,1,2,\ldots \quad (4)$$

In particular, the incomplete beta function, related to the Gauss hypergeometric $\varphi(a,c;z)$, is defined by

$$\varphi(a,c;z)=F_c^{a,1}(z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}z^{n+1} \quad (5)$$

where $z\in U$ and $c\neq 0,1,2,\ldots$.

Throughout this paper, let $G(z)=\phi_1(z)+\overline{\phi_2(z)}$ be a function where $\phi_1(z)$ and $\phi_2(z)$ are the hypergeometric functions defined by

$$\phi_1(z):=zF_{c_1}^{a_1,b_1}(z)=z+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}z^n \quad (6)$$

and

$$\phi_2(z):=F_{c_2}^{a_2,b_2}(z)-1=\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}z^n \quad (7)$$

where $|a_2b_2|<|c_2|$. The following lemma is needed to prove our results.

Lemma 1. (Ahuja, 2008) If $a,b,c>0$, then

$$\sum_{n=1}^{\infty}n\frac{(a)_n(b)_n}{(c)_n(1)_n}=\frac{ab}{c-a-b-1}F_c^{a,b}(1) \quad (8)$$

if $c>a+b+1$

$$\sum_{n=1}^{\infty}n^2\frac{(a)_n(b)_n}{(c)_n(1)_n}=\left[\frac{(a)_2(b)_2}{(c-a-b-2)_2}+\frac{ab}{c-a-b-1}\right]F_c^{a,b}(1) \quad (9)$$

if $c>a+b+2$.

Based on the study in (Yalçin & Öztürk, 2004), for $\alpha\geq 0$ and $0\leq\beta<1$, we define a class $HP(\alpha,\beta)$ of harmonic functions of the form (1) satisfying the condition

$$\operatorname{Re}\{\alpha z[h''(z)+g''(z)]+[h'(z)+g'(z)]\}>\beta.$$

Lemma 2. If $f=h+\overline{g}$ is given by (1) and

$$\sum_{n=1}^{\infty}n[\alpha(n-1)+1](|A_n|+|B_n|)\leq 2-\beta \quad (10)$$

where $0\leq|B_1|<1-\beta, A_1=1, \alpha\geq 0$ and $0\leq\beta<1$, then f is harmonic univalent and sense preserving in U and $f\in HP(\alpha,\beta)$.

Proof. The proof of this lemma is on lines similar to the proof of Theorem 2.1 in (Yalçin & Öztürk, 2004).

II. MAIN RESULTS

Theorem 1. If $a_j, b_j > 0$ and $c_j > a_j + b_j + 2$ for $j = 1, 2,$ then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ to be harmonic univalent in U and $G \in HP(\alpha, \beta),$ is that

$$\left[\frac{\alpha(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2}+\frac{a_1b_1(2\alpha+1)}{c_1-a_1-b_1-1}+1\right]F_{c_1}^{a_1,b_1}(1)$$
$$+\left[\frac{\alpha(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2}+\frac{a_2b_2}{c_2-a_2-b_2-1}\right]F_{c_2}^{a_2,b_2}(1)$$
$$\leq 2-\beta \qquad (11)$$

where $\alpha \geq 0$ and $0 \leq \beta < 1.$

Proof. When the condition (11) holds for the coefficients of $G = \phi_1 + \overline{\phi_2}$, by using (10) it is enough to prove that

$$\sum_{n=1}^{\infty} n(\alpha(n-1)+1)\left[\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\right]$$
$$\leq 2-\beta. \qquad (12)$$

Write the left side of equality (12) as

$$\alpha\sum_{n=1}^{\infty} n(n-1)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}$$
$$+\alpha\sum_{n=1}^{\infty} n(n-1)\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$+\sum_{n=1}^{\infty} n\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\sum_{n=1}^{\infty} n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$=\alpha\sum_{n=1}^{\infty}[(n-1)^2+(n-1)]\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}$$
$$+\alpha\sum_{n=1}^{\infty}(n^2-n)\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$+\sum_{n=1}^{\infty}(n-1+1)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\sum_{n=1}^{\infty} n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$=\alpha\sum_{n=1}^{\infty} n^2\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n}+(\alpha+1)\sum_{n=1}^{\infty} n\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n}$$
$$+\sum_{n=0}^{\infty}\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n}+\alpha\sum_{n=1}^{\infty} n^2\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$-(\alpha-1)\sum_{n=1}^{\infty} n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$

$$=\alpha\left[\frac{(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2}+\frac{a_1b_1}{c_1-a_1-b_1-1}\right]F_{c_1}^{a_1,b_1}(1)$$
$$+(\alpha+1)\frac{a_1b_1}{c_1-a_1-b_1-1}F_{c_1}^{a_1,b_1}(1)+F_{c_1}^{a_1,b_1}(1)$$
$$+\alpha\left[\frac{(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2}+\frac{a_2b_2}{c_2-a_2-b_2-1}\right]F_{c_2}^{a_2,b_2}(1)$$
$$-(\alpha-1)\frac{a_2b_2}{c_2-a_2-b_2-1}F_{c_2}^{a_2,b_2}(1),$$

by an application of equation (8) and (9). This yield (11). In order to prove that G is locally univalent and sense-preserving in U, it is sufficient to show that $|\phi'_1(z)| > |\phi'_2(z)|,$

$$|\phi'_1(z)| = \left|1+\sum_{n=2}^{\infty} n\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^{n-1}\right|$$

$$> 1-\sum_{n=2}^{\infty}(n-1)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}$$

$$-\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}$$

$$=1-\frac{a_1b_1}{c_1}\sum_{n=1}^{\infty}\frac{(a_1+1)_{n-1}(b_1+1)_{n-1}}{(c_1+1)_{n-1}(1)_{n-1}}$$

$$-\sum_{n=1}^{\infty}\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n}$$

$$\geq 2-\beta-\left[\begin{array}{c}\dfrac{\alpha(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2}\\ +\dfrac{a_1b_1(2\alpha+1)}{c_1-a_1-b_1-1}+1\end{array}\right]$$

$$\times F_{c_1}^{a_1,b_1}(1)$$

$$\geq\left[\begin{array}{c}\dfrac{\alpha(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2}\\ +\dfrac{a_2b_2}{c_2-a_2-b_2-1}\end{array}\right]F_{c_2}^{a_2,b_2}(1)$$

$$\geq\frac{a_2b_2}{c_2}\frac{\Gamma(c_2+1)\Gamma(c_2-a_2-b_2-1)}{\Gamma(c_2-a_2)\Gamma(c_2-b_2)}$$

$$=\sum_{n=0}^{\infty}\frac{(a_2)_{n+1}(b_2)_{n+1}}{(c_2)_{n+1}(1)_n}$$

$$\geq\left|\sum_{n=1}^{\infty} n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} z^{n-1}\right| = |\phi'_2(z)|.$$

In fact, for $|z_1|\leq|z_2|<1,$ we have

$$|G(z_1)-G(z_2)|$$

$$\geq|\phi_1(z_1)-\phi_1(z_2)|-|\phi_2(z_1)-\phi_2(z_2)|$$

$$=\left|(z_1-z_2)+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}(z_1^n-z_2^n)\right|$$

$$-\left|\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}(z_1^n-z_2^n)\right|$$

$$\geq|z_1-z_2|\left[\begin{array}{c}1-\dfrac{a_2b_2}{c_2}\\ -\displaystyle\sum_{n=2}^{\infty} n\left(\begin{array}{c}\dfrac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}\\ +\dfrac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\end{array}\right)|z_2|^{n-1}\end{array}\right]$$

$$=|z_1-z_2|$$

$$\times\left[\begin{array}{c}2-\beta\\ -\displaystyle\sum_{n=1}^{\infty} n(\alpha(n-1)+1)\left[\begin{array}{c}\dfrac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}\\ +\dfrac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\end{array}\right]\end{array}\right]$$

In view of (12), $|G(z_1)-G(z_2)|\geq 0$ which shows that G is univalent in U. This completes the proof.

Denote by $HT(\alpha,\beta)=HP(\alpha,\beta)\bigcap T_H$ where T_H [16], is the class of harmonic functions $f=h+\bar{g}$ of the form

$$h(z)=z-\sum_{n=2}^{\infty}A_nz^n \text{ and } g(z)=-\sum_{n=1}^{\infty}B_nz^n \quad (13)$$

where $A_n, B_n\geq 0$ for $n=1,2,\ldots$ and $B_1<1.$

Lemma 3. If $f=h+\bar{g}$ is given by (13), then $f\in HT(\alpha,\beta)$ if and only if

$$\sum_{n=1}^{\infty} n[\alpha(n-1)+1](A_n+B_n)\leq 2-\beta$$

where $\alpha\geq 0,\ 0\leq\beta<1,\ A_1=1$ and $0\leq B_1<1-\beta.$

The sufficiency of this result is from Lemma 2 and the proof of necessity is on lines similar to the proof of Theorem 2.2 in (Yalçin & Öztürk, 2004). Define

$$G_1(z) = z\left(2 - \frac{\phi_1(z)}{z}\right) - \overline{\phi_2(z)}$$
$$= z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^n$$
$$-\overline{\sum_{n=1}^{\infty} \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} z^n}$$

on using (6) and (7). Clearly $G_1 \in T_H$. **Theorem 2.** Let $\alpha \geq 0, 0 \leq \beta < 1, a_j, b_j > 0,\ c_j > a_j + b_j + 2,$ for $j = 1, 2$ and $a_2 b_2 < c_2$. G_1 is in $HT(\alpha, \beta)$ if and only if (11) holds.

Proof. If $G_1 \in HT(\alpha, \beta)$, then G_1 satisfies (12) by Lemma 3 and hence (11) holds.

Theorem 3. Let $0 \leq \beta < 1, a_j, b_j > 0, c_j > a_j + b_j + 1,$ for $j = 1, 2$ and $a_2 b_2 < c_2$. A necessary and sufficient condition such that $f * (\phi_1 + \overline{\phi_2}) \in HT(\alpha, \beta)$ for $f \in HT(\alpha, \beta)$ is that

$$F_{c_1}^{a_1,b_1}(1) + F_{c_2}^{a_2,b_2}(1) \leq 3 \tag{14}$$

where ϕ_1, ϕ_2 are as defined, respectively, by (6) and (7).

Proof. Let $f = h + \overline{g} \in HT(\alpha, \beta)$, where h and g are given by (13). Then

$$(f * (\phi_1 + \overline{\phi_2}))(z) = h(z) * \phi_1(z) + \overline{g(z) * \phi_2(z)}$$
$$= z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n z^n$$
$$-\overline{\sum_{n=1}^{\infty} \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} B_n z^n}.$$

In view of Lemma 3, we need to prove that $(f * (\phi_1 + \overline{\phi_2})) \in HT(\alpha, \beta)$ if and only if

$$\sum_{n=1}^{\infty} n(\alpha(n-1)+1) \left[\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n + \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} B_n \right]$$
$$\leq 2 - \beta. \tag{15}$$

As an application of Lemma 3, we have

$$A_n \leq \frac{1-\beta}{n(\alpha(n-1)+1)}, n = 2, 3, \ldots$$

and

$$B_n \leq \frac{1-\beta}{n(\alpha(n-1)+1)}, n = 1, 2, \ldots.$$

Therefore, the left side of (15) is bounded above by

$$\sum_{n=2}^{\infty} (1-\beta) \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} (1-\beta) \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$= (1-\beta) \left[\sum_{n=1}^{\infty} \frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} + \sum_{n=1}^{\infty} \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \right]$$
$$= (1-\beta)[F_{c_1}^{a_1,b_1}(1) + F_{c_2}^{a_2,b_2}(1) - 2].$$

The last expression is bounded above by $(1-\beta)$ if and only if (14) is satisfied. This proves (15) and the results follows.

Theorem 4. If $a_j, b_j > 0$ and $c_j > a_j + b_j + 1$ for $j = 1, 2$, then a sufficient condition for a function

$$G_2(z) = \int_0^z F_{c_1}^{a_1,b_1}(t)dt + \overline{\int_0^z [F_{c_2}^{a_2,b_2}(t) - 1]dt}$$

to be in $HP(\alpha, \beta)$ is that

$$\left(\frac{\alpha(a_1 b_1)}{c_1 - a_1 - b_1 - 1} + 1 \right) F_{c_1}^{a_1,b_1}(1)$$
$$+ \left(\frac{\alpha(a_2 b_2)}{c_2 - a_2 - b_2 - 1} + 1 \right) F_{c_2}^{a_2,b_2}(1)$$
$$\leq 3 - \beta$$

where $\alpha \geq 0$ and $0 \leq \beta < 1$.

Proof. In view of Lemma 2, the function

$$G_2(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} z^n$$
$$+\overline{\sum_{n=2}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} z^n}$$

is in $HP(\alpha, \beta)$ if

$$\sum_{n=2}^{\infty} n(\alpha(n-1)+1) \left[\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} + \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} \right]$$
$$\leq 1 - \beta. \tag{16}$$

By a simple computation we obtain

$$\sum_{n=2}^{\infty} n(\alpha(n-1)+1)\left[\begin{array}{c}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n}\\ +\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n}\end{array}\right]$$

$$=\sum_{n=1}^{\infty}(\alpha n+1)\left[\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n}+\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\right].$$

The result follows from an application of Lemma 1.

Theorem 5. If $a_1, b_1 > -1, c_1 > 0, a_1b_1 < 0,$ $a_2 > 0, b_2 > 0,$ and $c_j > a_j + b_j + 2, j = 1, 2,$ then

$$G_3(z)=\int_0^z F_{c_1}^{a_1,b_1}(t)dt-\overline{\int_0^z [F_{c_2}^{a_2,b_2}(t)-1]dt}$$

to be in $HT(\alpha,\beta)$ if and only if

$$\left(\frac{\alpha(a_1b_1)}{c_1-a_1-b_1-1}+1\right)F_{c_1}^{a_1,b_1}(1)$$
$$-\left(\frac{\alpha(a_2b_2)}{c_2-a_2-b_2-1}+1\right)F_{c_2}^{a_2,b_2}(1)+1\geq\beta$$

where $\alpha \geq 0$ and $0 \leq \beta < 1.$

Proof. We write

$$G_3(z)=z-\frac{|a_1b_1|}{c_1}\sum_{n=2}^{\infty}\frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_n}z^n$$
$$-\overline{\sum_{n=2}^{\infty}\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n}z^n}.$$

In view of Lemma 3 it is sufficient to show that

$$\sum_{n=2}^{\infty} n(\alpha(n-1)+1)\left[\begin{array}{c}\frac{|a_1b_1|}{c_1}\frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_n}\\ +\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n}\end{array}\right]$$
$$\leq 1-\beta. \tag{17}$$

By a routine computation (17) can be written as

$$\alpha\sum_{n=1}^{\infty}\frac{|a_1b_1|}{c_1}\frac{(a_1+1)_{n-1}(b_1+1)_{n-1}}{(c_1+1)_{n-1}(1)_{n-1}}$$
$$+\alpha\sum_{n=1}^{\infty}n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$+\sum_{n=1}^{\infty}\frac{|a_1b_1|}{c_1}\frac{(a_1+1)_{n-1}(b_1+1)_{n-1}}{(c_1+1)_{n-1}(1)_n}+\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$\leq(1-\beta).$$

Or equivalently

$$\alpha\sum_{n=0}^{\infty}\frac{(a_1+1)_n(b_1+1)_n}{(c_1+1)_n(1)_n}+\frac{\alpha c_1}{|a_1b_1|}\sum_{n=1}^{\infty}n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$+\sum_{n=0}^{\infty}\frac{(a_1+1)_n(b_1+1)_n}{(c_1+1)_n(1)_{n+1}}+\frac{c_1}{|a_1b_1|}\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$\leq\frac{c_1(1-\beta)}{|a_1b_1|}.$$

But, this is equivalent to

$$\frac{\alpha c_1}{a_1b_1}\sum_{n=1}^{\infty}n\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n}+\frac{\alpha c_1}{|a_1b_1|}\sum_{n=1}^{\infty}n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$+\frac{c_1}{a_1b_1}\sum_{n=1}^{\infty}\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n}+\frac{c_1}{|a_1b_1|}\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$\leq\frac{c_1(1-\beta)}{|a_1b_1|},$$

which yields

$$\left(\frac{\alpha(a_1b_1)}{c_1-a_1-b_1-1}+1\right)F_{c_1}^{a_1,b_1}(1)$$
$$-\left(\frac{\alpha(a_2b_2)}{c_2-a_2-b_2-1}+1\right)F_{c_2}^{a_2,b_2}(1)\geq -1+\beta.$$

This completes the proof.

In particular, the results parallel to Theorems 3, 6, 7 and 8 may also be obtained for the incomplete beta function $\varphi(a,c;z)$ as defined by (4) and (5). Let

$$\phi_1(z)=\varphi(a_1,c_1;z)=z+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}}{(c_1)_{n-1}}z^n,$$

and

$$\phi_2(z)=\frac{1}{z}\varphi(a_2,c_2;z)-1=\sum_{n=1}^{\infty}\frac{(a_2)_n}{(c_2)_n}z^n$$

where $|a_2|<|c_2|$. Making use of

$$F_{c_1}^{a_1,1}(1)=\frac{c_1-1}{c_1-a_1-1} \text{ and } F_{c_2}^{a_2,1}(1)-1=\frac{a_2}{c_2-a_2-1}$$

the following theorems are obtained.

Theorem 6. If $a_j > 0$ and $c_j > a_j + 3$ for $j = 1, 2,$ then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ to be harmonic univalent in U with $\phi_1 + \overline{\phi_2} \in HP(\alpha, \beta),$ is that

$$\left[\frac{2\alpha(a_1)_2}{(c_1-a_1-3)_2}+\frac{2\alpha a_1+c_1-2}{c_1-a_1-2}\right]\frac{c_1-1}{c_1-a_1-1}$$
$$+\left[\frac{2\alpha(a_2)_2}{(c_2-a_2-3)_2}+\frac{a_2}{c_2-a_2-2}\right]\frac{c_2-1}{c_2-a_2-1}$$
$$\leq 2-\beta \qquad (18)$$

where $\alpha \geq 0$ and $0 \leq \beta < 1$.

Note that the condition (18) is necessary and sufficient for $G=\phi_1+\overline{\phi_2}$ to be in $HT(\alpha,\beta)$.

Theorem 7. Let $0 \leq \beta < 1, a_j > 0,\ c_j > a_j + 2,$ for $j=1,2$ and $a_2 < c_2$. A necessary and sufficient condition such that $f*(\phi_1+\overline{\phi_2}) \in HT(\alpha,\beta)$ for $f \in HT(\alpha,\beta)$ is that

$$\frac{c_1-1}{c_1-a_1-1}+\frac{c_2-1}{c_2-a_2-1} \leq 3-\beta.$$

Theorem 8. If $a_j > 0$ and $c_j > a_j + 2$ for $j=1,2$, then sufficient condition for

$$\int_0^z \varphi(a_1,c_1;t)dt+\overline{\int_0^z [\varphi(a_2,c_2;t)-1]dt}$$

is in $HP(\alpha,\beta)$ is

$$\left(\frac{\alpha a_1}{c_1-a_1-2}+1\right)\frac{c_1-1}{c_1-a_1-1}+\left(\frac{\alpha a_2}{c_2-a_2-2}+1\right)\frac{c_2-1}{c_2-a_2-1}$$
$$\leq 3-\beta$$

where $\alpha \geq 0$ and $0 \leq \beta < 1$.

Theorem 9. If $a_1 > -1, c_1 > 0, a_1 < 0, a_2 > 0,$ and $c_j > a_j + 3,\ j=1,2,$ then

$$\int_0^z \varphi(a_1,c_1;t)dt-\overline{\int_0^z [\varphi(a_2,c_2;t)-1]dt}$$

is in $HT(\alpha,\beta)$ if and only if

$$\left(\frac{\alpha a_1}{c_1-a_1-2}+1\right)\frac{c_1-1}{c_1-a_1-1}-\left(\frac{\alpha a_2}{c_2-a_2-2}+1\right)\frac{c_2-1}{c_2-a_2-1}+1$$
$$\geq \beta$$

where $\alpha \geq 0$ and $0 \leq \beta < 1$.

Note that Theorems 3.1 and 3.5 in (Al-Khal & Al-Kharsani, 2006) are respectively obtained from Theorem 8 and Theorem 4.

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