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MHD Casson Fluid Flow over a Non-Linear Stretching/Shrinking Sheet with Suction and Slip Boundary Condition

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A problem of steady laminar magnetohydrodynamic (MHD) Casson flow over a non-linear stretching/shrinking sheet with suction and slip boundary condition is considered. The governing nonlinear partial differential equations are transformed into a system of nonlinear ordinary differential equations using a similarity transformation. The transformed ordinary differential equations are then solved numerically using the bvp4c function in MATLAB software. Dual solutions are found for a certain range of the governing parameter. The effects of the suction parameter and the slip boundary condition on the skin friction and heat transfer coefficients as well as the velocity and temperature profiles are presented and discussed.

Keywords: MHD, Casson, stretching/shrinking, dual solutions, slip

I. INTRODUCTION

The study of non-Newtonian fluids is very important due to its engineering and industrial process such as magnetic materials processing, magnetohydrodynamic electrical power generation, and paper production. Casson fluid is another type of non-Newtonian fluid which exhibits yield stress (Das & Batra 1993).

The boundary layer flow of non-Newtonian fluid has attained a considerable amount of attention in the past few decades due to its importance in engineering area. Sakiadis (1961) was the first to consider the problem of boundary layer flow over a stretching plate. Later, Mukhopadhyay (2013) extended Sakiadis (1961)'s paper by considering the problem on non-Newtonian fluids. Recently, Haldar et al. (2018) studied the heat transfer effect on Casson fluid flow over shrinking sheet with convective boundary condition.

Sarpkaya (1961) was the first to consider the magnetohydrodynamic (MHD) flow on non-Newtonian fluid. The non-Newtonian fluid motion with heat transfer in the existence of a transverse magnetic field was studied by Bestman (1985), which then extended by Eldabe et al. (2001) and Attia and Sayed-Ahmed (2006). Recently, Pal & Roy (2018) studied the Lie Group transformation on MHD double-diffusion convection of Casson fluid and found that the physical parameters have significant influence on the flow velocity and surface shear stress.

The non-Newtonian fluid flow in a microchannel with wall slip was explained by Koo & Kleinstreuer (2004). Furthermore, the effect of slip condition on magnetohydrodynamic flow of Casson fluid with Newtonian heating was described by Ullah et al. (2017). They observed that the slip parameter enhances the shear stress on the fluid but reduces the velocity of the Casson fluid. Most recently, Nagendrama et al. (2018) examined the three-dimensional magnetohydrodynamic Casson fluid flow across a slandering sheet with multiple slips condition.

In this study, we extend the research done by Ullah et al. (2017) by incorporating suction and both stretching and shrinking surfaces to solve the MHD Casson fluid flow with

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the effects of Newtonian heating and slip boundary condition. The governing nonlinear partial differential equations are converted to the system of ordinary differential equation using appropriate similarity transformation. We intend to solve the problem numerically and find the dual solutions by using the bvp4c function in MATLAB (Kierzenka & Shampine 2001).

II. PROBLEM FORMULATION

We consider the steady two dimensional and incompressible free convectional flow of Casson fluid over a non-linearly stretching or shrinking sheet saturated in a porous medium under the influence of magnetic field and wall mass transfer, as shown in Figure 1.The Newtonian heating and slip condition are also taken into account. The sheet is stretched/shrinked along the $x-$axis with the velocity of $ku_w(x) = kcx^n$, where c is a constant while k is a stretching $(k > 0)$ or shrinking $(k < 0)$ parameter and $y-$direction is taken for fluid flow with the origin fixed. It should be mentioned that $n=1$ corresponds to the linear stretching/shrinking case while $n \neq 1$ represents the non-linear case.

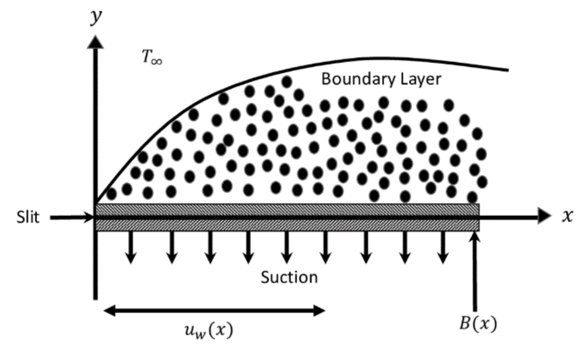

Figure 1. Sketch of physical problem

A non-uniform transverse magnetic field of strength B_0 is applied normal to the sheet. T_∞ denotes the ambient fluid temperature. The induced magnetic field is neglected due to small magnetic Reynolds number. The rheological equation for incompressible flow of Casson fluid is given by the following (Bhattacharyya, 2013),

$$\tau_{ij} = \begin{cases} 2\left(\mu_B + p_y/\sqrt{2\pi}\right)e_{ij}, & \pi > \pi_c \\ 2\left(\mu_B + p_y/\sqrt{2\pi_c}\right), & \pi < \pi_c \end{cases}$$

where $\pi = e_{ij}e_{ij}$ and e_{ij} are the $(i, j)-$ th component of the deformation rate, π is the product of the component of deformation rate with itself, π_c is a critical value of this product based on the non-Newtonian model, μ_B is the plastic dynamic viscosity of the non-Newtonian fluid and p_y is the yield stress of the fluid. Under these assumptions, the basic boundary layer equations of continuity, momentum and energy can be expressed as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} - \left(\frac{\sigma B^2(x)}{\rho} + \frac{\nu\phi}{k}\right)u + g\beta_T\left(T - T_\infty\right), \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

while the corresponding boundary conditions for the problem can be written as

$$\begin{aligned} &u = kcx^n + N_1\nu\left(1 + \frac{1}{\beta}\right)\frac{\partial u}{\partial u}, \\ &v = v_w,\ \frac{\partial T}{\partial y} = h_s T \ \text{at}\ y = 0, \\ &u \to 0,\ T \to T_\infty \ \text{as}\ y \to \infty, \end{aligned} \tag{4}$$

where u and v are the velocity components along the $x-$ and $y-$axes, respectively, ρ is the fluid density, ν is kinematic viscosity, β is the Casson fluid parameter, σ is the fluid's electrical conductivity, ϕ is the porosity, k is the permeability of porous medium, g is the gravitational acceleration, β_T is the volumetric coefficient of thermal expansion, T is the fluid temperature, and α is the thermal diffusivity of the Casson fluid. Furthermore, $B(x) = B_0x^{(n-1)/2}$ is the magnetic field with constant magnetic strength B_0, $N_1(x) = Nx^{-(n-1)/2}$ is the velocity of slip factor depends on x, while $h_s = h_0cx^{(n-1)/2}$ represents the heat transfer parameter for Newtonian heating.

We look now for a similarity solution of Eqs. (1)-(3) subjected to the initial and boundary condition (4) of the following form (Ullah et al. 2017)

$$\psi = \sqrt{\frac{2\nu c x^{n+1}}{n+1}}\, f(\eta),\quad \theta(\eta) = \frac{T - T_\infty}{T_\infty},\qquad \eta = \sqrt{\frac{(n+1)c x^{n-1}}{2\nu}}\, y, \tag{5}$$

where the stream function ψ is defined by $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Using(5), Eq. (1)is satisfied, while Eqs. (2)and(3) are transformed into the following ordinary differential equations

$$\left(1 + \frac{1}{\beta}\right) f''' + f f'' - \frac{2n}{n+1} f'^2 - \left(M^2 + K\right) f' + \lambda\theta = 0, \tag{6}$$

$$\theta'' + \Pr f \theta' = 0, \tag{7}$$

along with the corresponding boundary conditions

$$f(0) = s,\quad f'(0) = k + \delta\left(1 + \frac{1}{\beta}\right) f''(0),\quad \theta'(0) = -\gamma\left(\theta(0) + 1\right),\quad f'(\eta) \to 0,\ \theta(\eta) \to 0 \text{ as } \eta \to \infty, \tag{8}$$

where primes denote derivatives with respect to η, k is stretching $(k > 0)$ or shrinking $(k < 0)$ parameter and $s = -v_w \Big/ \left(\frac{n+1}{2}\sqrt{\frac{2\nu c x^{n-1}}{n+1}}\right)$ is the mass transfer parameter with $s > 0$ denotes suction, while $s < 0$ refer to injection and $s = 0$ is only applicable for non-permeable surface. Other parameters are magnetic parameter $M^2 = 2\sigma B_0^2/\left(\rho c(n+1)\right)$, porosity parameter $K = 2\nu\phi x/\left(k(n+1)c x^n\right)$, Reynolds number $\mathrm{Re}_x = c x^{n+1}/\nu$, local Grashof number $Gr_x = 2g\beta_T T_\infty x^3/\left(\nu^2(n+1)\right)$, Prandtl number $Pr = \nu/\alpha$, thermal buoyancy parameter $\lambda = Gr_x/Re_x^2$, Newtonian heating parameter $\gamma = h_0\sqrt{2\nu/c(n+1)}$ and slip parameter $\delta = N\sqrt{(n+1)c\nu/2}$.

The main physical quantities of interest are the reduced skin friction coefficient in $x-$ direction and local Nusselt number which are defined as the following

$$Cf_x = \frac{\tau_w}{\rho u_w^2},\quad Nu_x = \frac{x q_w}{\alpha(T_w - T_\infty)}, \tag{9}$$

where τ_w denotes the wall skin friction, while q_w is the wall heat flux, which are given by

$$\tau_w = \mu_B\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)_{y=0},\qquad q_w = -\alpha\left(\frac{\partial T}{\partial y}\right)_{y=0}, \tag{10}$$

with μ_B and α are dynamics viscosity and thermal diffusivity for the fluid, respectively. Using the similarity solution in (5), we obtain the following expressions

$$(Re)^{1/2} Cf_x\sqrt{\frac{2}{n+1}} = \left(1 + \frac{1}{\beta}\right) f''(0),\qquad (Re_x)^{-1/2} Nu_x\sqrt{\frac{\nu}{n+1}} = -\theta'(0). \tag{11}$$

III. RESULTS AND DISCUSSIONS

The system of equations (6)-(8) are solved numerically using the MATLAB boundary value problem solver called bvp4c. We set the relative error tolerance to 10^{-5} throughout the numerical computation. The following Table 1shows the excellent comparison of the present study with the previous results obtained by Ullah et al. (2017), which proves that the present method is accurate, and the results are correct.

Table 1. Comparison of numerical results of $-f''(0)$ and $-\theta'(0)$ for some values of n with $k = 1$, $\beta = 10^8$, $\gamma = 10^4$ and $M = K = \lambda = \delta = s = 0$

n	Ullah et al. (2017)		Present	
	$-f''(0)$	$-\theta'(0)$	$-f''(0)$	$-\theta'(0)$
0.2	0.7668	0.6102	0.7668	0.6103
0.5	0.8896	0.5949	0.8896	0.5953
3	1.1468	0.5647	1.1486	0.5647
10	1.2349	0.5549	1.2349	0.5550

Figures 2 and 4 illustrate the variations of skin friction coefficients $f''(0)$ and local Nusselt numbers $-q'(0)$ with stretching/shrinking parameter k for some values of suction s when $b = 0.4$, $n = 1$, $M = 0.3$, $K = 0.1$, $l = 0.6$, $Pr = 0.7$, $d = 0.2$ and $g = 0.4$, while Figures 6 and 7 illustrate the variations of $f''(0)$ and $-q'(0)$ with k for some values of slip parameter d when $b = 0.7$, $n = 10$, $M = 0.3$, $K = 0.1$, $l = 0.6$, $Pr = 0.7$, $s = 3$ and $g = 0.4$. Based on these figures, it is worth to mention that two solutions are visible for both shrinking $(k < 0)$ and stretching $(k > 0)$ cases, whereas the upper branch solution and lower branch solution are presented by solid and dotted lines, respectively. However both solutions only exist up to k_c, where k_c is the critical value of k, and no solution can be found for $k < k_c$. In this zone, the boundary layer begins to separate from the surface, hence the boundary layer approximation method is no longer applicable and one needs to consider the full Navier-Stokes equations.

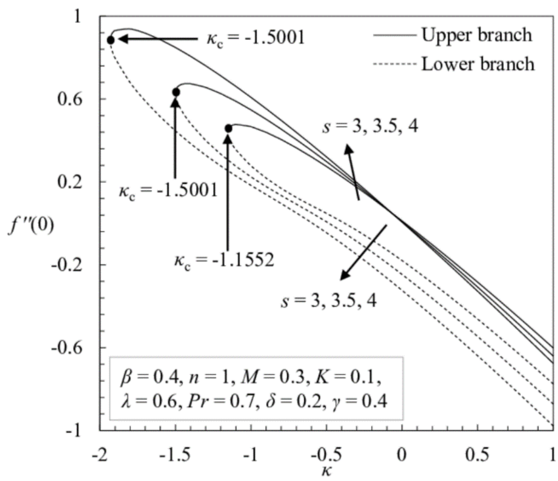

Figure 2. Variations of $f''(0)$ with k for some values of s

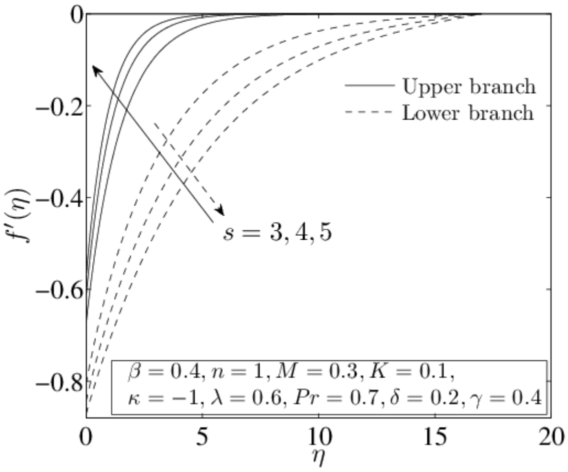

Figure 3. Velocity profiles $f'(h)$ for different values of s

Figures 2 and 3 display the influence of suction parameter $s(> 0)$ on skin friction coefficients $f''(0)$ and velocity profiles $f'(h)$, respectively. It seems that the values of $f''(0)$ are gradually increasing with the increase of s. The existence of suction causes the reduction in boundary layer thickness (as shown in Figure 3), which eventually force the fluid flow to decelerate and move closer to the boundary and hence enhance the velocity gradient at the surface of the sheet. It can also be noticed that $f''(0)$ becomes larger when the sheet is shrinking. The slow motion of the fluid along the shrinking sheet forms the opposing flow, which results in the separation and therefore enhancing the skin friction coefficients.

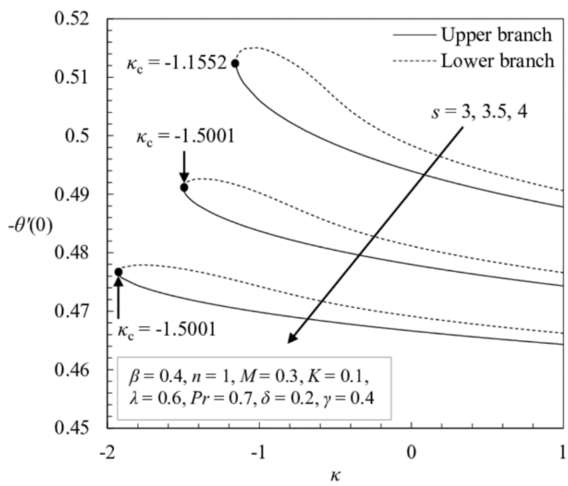

Figure 4. Variations of $-q'(0)$ with k for some values of s

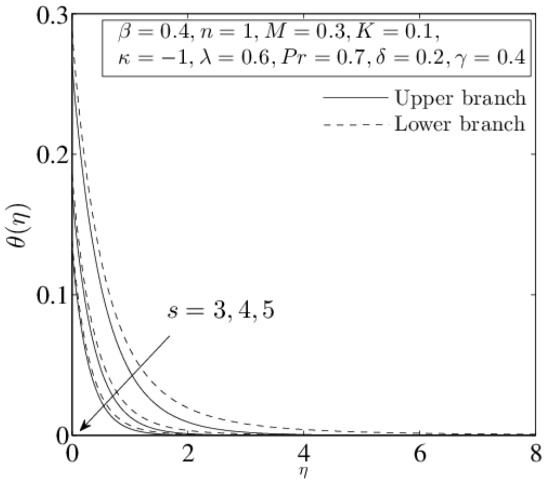

Figure 5. Temperature profiles $q(h)$ for different values of s

The effects of suction on local Nusselt number $-q¢(0)$ and temperature profile $q(h)$ are presented in Figures 4and 5, respectively. It can be seen in Figure 4that the values of $-q¢(0)$ decrease with the increase of s. It is also observable that the rate of heat transfer is higher as the sheet is shrinking. Meanwhile, Figure 5 shows the decrement of thermal boundary layer thickness as s increases. As the thickness getting thinner, the temperature gradient increases and hence increases the heat transfer rate.

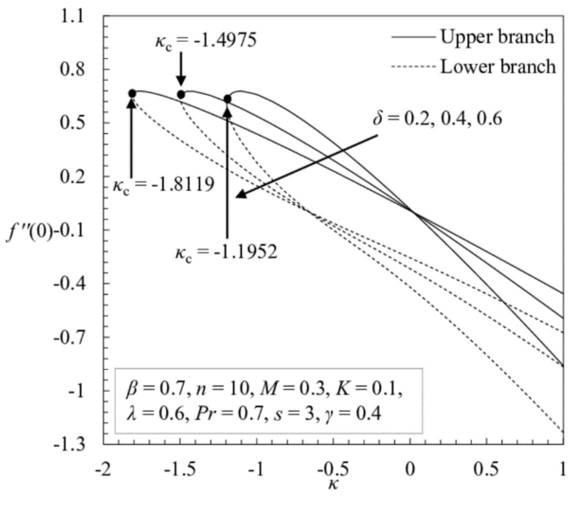

Figure 6. Variations of $f¢¢(0)$ with k for some values of d

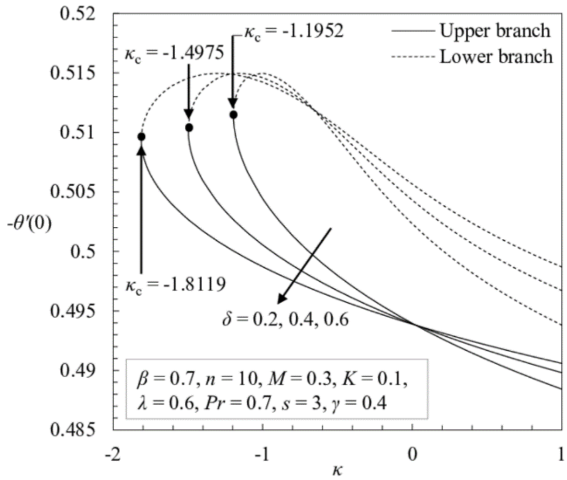

Figure 7. Variations of $-q¢(0)$ with k for some values of d

Figures 6 and 7 display the variations of $f¢¢(0)$ and $-q¢(0)$ with k for some values of the slip parameter d when $b = 0.7,\ n = 10,\ M = 0.3,\ K = 0.1,\ l = 0.6,\ Pr = 0.7,\ s = 3$ and $g = 0.4$. Both figures show the decreasing values of $f¢¢(0)$ and $-q¢(0)$ with the increase of d. Increasing slip at the boundary decreases the wall shear stress, thus reducing the vorticity generated for shrinking velocity due to the weakening fluid adhesion strength. However for larger shrinking velocity, the vorticity remained restricted within the boundary layer by the aid of suction, hence the possibility of steady solution for some large values of $|k|$ (Bhattacharyya 2011). In additional, it also shows that the inclusion of slip can greatly change the wall drag force.

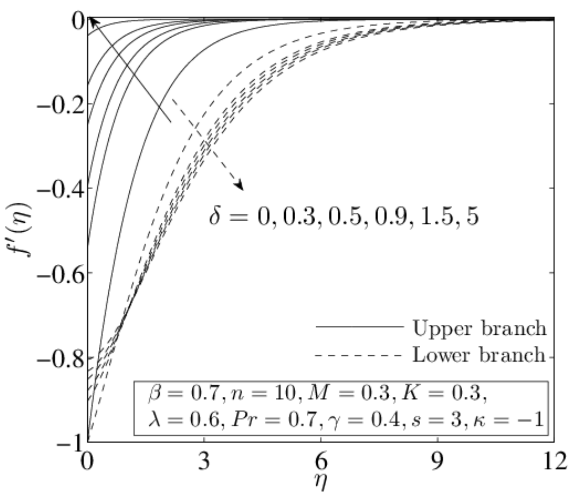

Figure 8. Velocity profiles $f¢(h)$ for different values of d

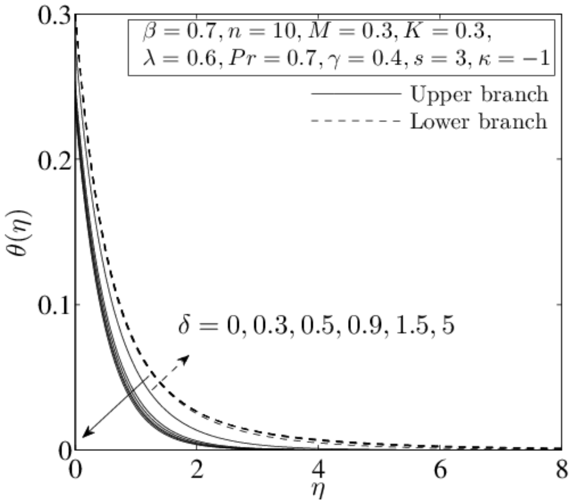

Figure 9.Temperature profiles $q(h)$ for different values of d

Figures 8 and 9 illustrate the velocity profiles $f ¢(h)$ and temperature profiles $q(h)$, respectively for different values of d for the case of shrinking sheet $(k = - 1)$. It is seen that for the given stretching/shrinking parameter, the boundary layer thickness for both solution branches becomes smaller with the increase of the slip parameters. Physically, this means the increasing slip allows more fluid to flow through the surface, thus reducing the boundary layer thickness.

IV. CONCLUSION

The problem of magnetohydrodynamic (MHD) Casson flow over a non-linear stretching/shrinking sheet with suction and slip boundary condition has been considered and solved numerically using the bvp4c function in MATLAB. The effects of suction and slip parameters to the skin friction coefficients, local Nusselt numbers and velocity and temperature profiles have been analysed and presented graphically. Multiple solutions are found for a certain range of the stretching/shrinking parameter. The suction parameter widens the range of stretching/shrinking parameter for which similarity solutions exist. The values of the skin friction coefficients are found to increase with the increase of suction and stretching/shrinking parameters, while decrease with the increase of slip parameters. However, the values of the local Nusselt numbers are found to decrease with the increase of suction and slip parameters, while increase with the increase of stretching/shrinking parameter. All profiles displayed that the boundary layer thickness for the second solution (lower branch) is larger than the first solution (upper branch).

V. ACKNOWLEDGEMENT

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