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Alternating Direction Implicit (ADI) Method for Solving Two Dimensional (2-D) Transient Heat Equation

Norzilah Abdul Halif[1] and Nursalasawati Rusli[2]

[1]*Centre of Excellence for Frontier Materials Research, School of Materials Engineering, University Malaysia Perlis, Malaysia, Taman Pertiwi Indah, Jalan Kangar-Alor Setar, Seriab, 01000 Kangar, Perlis.*
[2]*Institute of Engineering Mathematics, University Malaysia Perlis, Pauh Putra Campus, 02600, Arau, Perlis.*

Heat equation is widely used in engineering problem specifically for prediction of temperature distribution during heating or cooling of solid material in high temperature furnace. This paper presents the implementation of Alternating Direction Implicit (ADI) method to solve the two-dimensional (2-D) heat equation with Dirichlet boundary condition. The boundary condition, initial condition, space and time step are programmed and executed in MATLAB to approximate the temperature distribution within the studied solid material. ADI formulation is validated by the semi-analytical method and is proven to be used successfully to solve 2-D heat equation.

Keywords:2-D Transient Heat Equation; ADI; Dirichlet boundary condition

I. INTRODUCTION

The heat conduction equation is categorized as a parabolic partial differential equation (PDE) and generally can be solved analytically or numerically. Analytical methods, namely Laplace transform (Lawal *et al.*, 2015), conformal mapping (Fan *et al.*, 2013), and homotopy analysis (Mahalakshmi *et al.*, 2012) have been used to solve transient heat conduction. Yet, they were successfully resolved the equation, provided that the problems are highly simplified and has regular geometries. In contrast, numerical methods serve the practical solution to the real problems involving irregular shape and non-uniform thermal conditions (Cengel, 2003) and have shown an excellent accuracy (Sameti *et al.*, 2014).

To date, the reports that available in the literature are focusing on the estimation of temperature distribution inside the walking-beam type reheating furnace or inside the heated metal slab or billet. Four numerical methods that commonly used are Finite Element Method (FEM), Boundary Element Method (BEM), Finite Volume Method (FVM) and Finite Difference Method (FDM).

FDM has been used extensively to estimate the temperature profile in metal slab (Banerjee *et al.*, 2004) or billet (Dubey *et al.*, 2012) or during the skid cooling (Abuluwefa & Alfantazi, 2014) when reheating furnace is operated. Validation of the numerical solution is frequently compared with the analytical solution (Dubey & Srinivasan, 2013) or based on the measured temperature from the experimental work (Abuluwefa & Alfantazi, 2014). Researchers also incorporated other factors during the numerical work such as temperature-dependent properties of furnaces (Chen *et al.*, 2005) and including the significances from the temperature variation to the formation of oxide on the surface of the metal (Anton *et al.*, 2002).

Alternating Direction Implicit (ADI) is recognized as simple and efficient method for solving 2-D parabolic problems particularly the heat equation. This method was originally proposed by Peaceman and Rachford in 1955 (Peaceman & Rachford, 1955). The main principle of the method is to break a 2-D problem into two 1-D problems

*Corresponding author's e-mail: norzilah@unimap.edu.my

solved by implicit schemes without forgoing the stability limitation (Li & Chen, 2008). Since then this method has been used widely in heat transfer analysis for engineering design.

In the present paper, one problem of transient 2-D heat equation is solved using ADI method and the approximated temperature is validated by semi-analytical solution.

The process of discretisation is presented thoroughly and simulated using MATLAB. The algorithm is believed can be extended to be used for various heat conduction problem involving Dirichlet boundary condition.

II. MATERIALS AND METHOD

A. Problem

A square solid material is cooled from its initial temperature of 30°F with boundary condition for each sides are fixed at 0°F (Bruh & Zyvoloski, 1974). as shown in Figure 1.

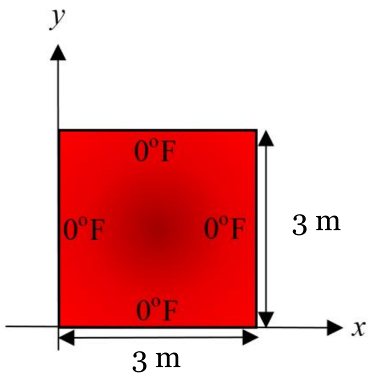

Figure 1. Schematic diagram of the problem

A transient 2-D heat conduction in the material can be mathematically expressed by [14]:

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) = \rho C_p \frac{\partial T}{\partial t} \qquad (1)$$

where the density, ρ and the heat capacity, C_p of the material are $1\mathrm{Ib_m/m^3}$ and $1\mathrm{Btu/Ib_m}$, respectively. The heat conductivity for both axis, k_x and k_y is $1.25\mathrm{Btu/hr.m^{\circ}F}$.The heat conduction is subjected to the following boundary and initial conditions as follows:

$$T(0,y,t)=T(x,0,t)=T(L_x,y,t)=T(x,L_y,t)=0 \qquad (2)$$

$$T(x,y,0)=30 \qquad (3)$$

where $\mathrm{L_x=3m}$ and $\mathrm{L_y=3m}$ are the length of the solution domain in x and y directions, respectively.

B. ADI Method

1. Discretisation of 2-D heat equation

The main principle of ADI method is solving the x -sweep implicitly and y sweep explicitly. First, the equation is discretised using forward differencing for the time derivative and central differencing for the space derivatives. The discretised equation is then producing two equations that will be solved sequentially from time level of n to $n+\frac{1}{2}$ and finally to $n+1$ as illustrated on the grid system in Figure 2.

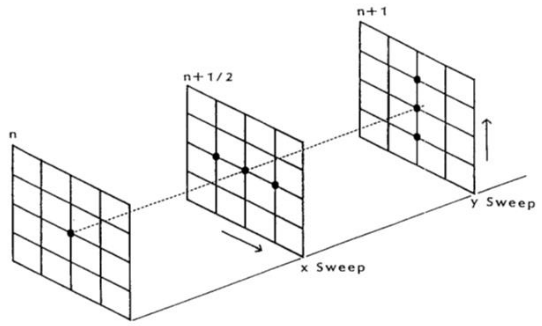

Figure 2. Illustration of the grid system for the ADI method [15]

For the time level of $n$to $n+1$, the Equation (1) is discretised into the following forms:

$$\frac{T_{i,j}^{n+\frac{1}{2}} - T_{i,j}^{n}}{\left(\frac{\Delta t}{2}\right)} = \alpha\left[\frac{T_{i+1}^{n+\frac{1}{2}} - 2T_{i,j}^{n+\frac{1}{2}} + T_{i-1,j}^{n+\frac{1}{2}}}{(\Delta x)^2} + \frac{T_{i+1}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}}{(\Delta y)^2}\right] \qquad (4)$$

And

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n+\frac{1}{2}}}{\left(\frac{\Delta t}{2}\right)} = \alpha\left[\frac{T_{i+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i-1,j}^{n+1}}{(\Delta x)^2} + \frac{T_{i+1}^{n+\frac{1}{2}} - 2T_{i,j}^{n+\frac{1}{2}} + T_{i-1,j}^{n+\frac{1}{2}}}{(\Delta y)^2}\right] \qquad (5)$$

Where Δt = time spacing; Δx = x spacing and Δy = y spacing. By substituting $d_2 = \frac{1}{2}d_x = \frac{1}{2}\alpha\frac{\Delta t}{(\Delta x)^2}$ into Equation (4), yields:

$$-d_1T_{i-1,j}^{n+\frac{1}{2}} + (2d_1+1)T_{i,j}^{n+\frac{1}{2}} - d_1T_{i+1,j}^{n+\frac{1}{2}} = d_2T_{i,j+1}^{n} + (1-2d_2)T_{i,j}^{n} + d_2T_{i,j-1}^{n} \quad (6)$$

by replacing $d_2 = \frac{1}{2}d_y = \frac{1}{2}\alpha\frac{\Delta t}{(\Delta y)^2}$, the Equation (5) becomes:

$$(1+2d_2)T_{i,j}^{n+1} - d_2T_{i,j+1}^{n+1} - d_2T_{i,j-1}^{n+1} = d_1T_{i+1,j}^{n+\frac{1}{2}} + (1-2d_1)T_{i,j}^{n+\frac{1}{2}} + d_1T_{i-1,j}^{n+\frac{1}{2}} \quad (7)$$

By introducing the coefficients as follows:

$a_1 = -d_1$,
$b_1 = 1+2d_1$,
$c_1 = -d_1$,

the Equation (6) becomes:

$$a_1T_{i-1,j}^{n+\frac{1}{2}} + b_1T_{i,j}^{n+\frac{1}{2}} + c_1T_{i+1,j}^{n+\frac{1}{2}} = d_2T_{i,j+1}^{n} + (1-2d_2)T_{i,j}^{n} + d_2T_{i,j-1}^{n} \quad (8)$$

The right-hand side of Equation (7) can be represented as

$$D_1 = d_2T_{i,j+1}^{n} + (1-2d_2)T_{i,j}^{n} + d_2T_{i,j-1}^{n} \quad (9)$$

In addition, by inserting the following coefficients into Equation (7):

$a_2 = -d_2$,
$b_2 = 1+2d_2$,
$c_2 = -d_2$,

produces:

$$a_2T_{i,j+1}^{n+1} + b_2T_{i,j}^{n+1} + c_2T_{i,j-1}^{n+1} = d_1T_{i+1,j}^{n+\frac{1}{2}} + (1-2d_1)T_{i,j}^{n+\frac{1}{2}} + d_1T_{i-1,j}^{n+\frac{1}{2}} \quad (10)$$

The right-hand side of Equation (10) can be represented as:

$$D_2 = d_1T_{i+1,j}^{n+\frac{1}{2}} + (1-2d_1)T_{i,j}^{n+\frac{1}{2}} + d_1T_{i-1,j}^{n+\frac{1}{2}} \quad (11)$$

D_1 and D_2 are the important elements in the matrix formulation of Equations (8) and (10) that need to be solved in the next step.

C. Development of Numerical Algorithm of ADI Method

Numerical algorithm of ADI for Equation (8) is developed for all grid points to produce set of linear algebraic equations. At initial condition, D_1 is the same at all grid points. Based on Equation (8), at $i = 2$ and $j = 2$:

$$a_1T_{1,2}^{n+\frac{1}{2}} + b_1T_{2,2}^{n+\frac{1}{2}} + c_1T_{3,2}^{n+\frac{1}{2}} = D_1 \quad (12)$$

Since $T_{1,2}^{n+\frac{1}{2}}$ is located at the boundary, the equation becomes:

$$b_1T_{2,2}^{n+\frac{1}{2}} + c_1T_{3,2}^{n+\frac{1}{2}} = D_1 - a_1T_{1,2}^{n+\frac{1}{2}} \quad (13)$$

At $i = IN-1$ and $j = 2$ (at the boundary), yields:

$$a_1T_{\text{IN}-2,2}^{n+\frac{1}{2}} + b_1T_{\text{IN}-1,2}^{n+\frac{1}{2}} = D_1 - c_1T_{\text{IN},2}^{n+\frac{1}{2}} \quad (14)$$

Where IN =Number of *i*, and JN = Number of *j*.

Therefore, the ADI algorithm for x -sweep can be written as:

$$\begin{pmatrix} b_1 & c_1 & 0 \\ a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ 0 & a_1 & b_1 \end{pmatrix}\begin{pmatrix} C_{2,2} \\ C_{3,2} \\ C_{\text{NI}-1,2} \end{pmatrix} = \begin{pmatrix} D_1 - a_1C_{1,2}^{n+\frac{1}{2}} \\ D_1 \\ D_1 \\ D_1 - c_1C_{\text{IN},2}^{n+\frac{1}{2}} \end{pmatrix} \quad (15)$$

Using the same step, the ADI algorithm for y -sweep can be presented as:

$$\begin{pmatrix} b_2 & c_2 & 0 \\ c_2 & b_2 & a_2 \\ c_2 & b_2 & a_2 \\ 0 & c_2 & b_2 \end{pmatrix}\begin{pmatrix} C_{2,2} \\ C_{2,3} \\ C_{2,4} \end{pmatrix} = \begin{pmatrix} D_{2,j=2} - c_2C_{2,1}^{n+1} \\ D_{2,j=2} \\ D_{2,j=3} \\ D_{2,j=IN-1} - a_2C_{2,IN}^{n+1} \end{pmatrix} \quad (16)$$

The tridiagonal system of equations (15) and (16) are then solved for all *i* and *j*.The approximated temperature will be validated with analytical method.

D. Semi-Analytical Solution

The semi-analytical solution for this problem is:

$$T(x,y,t)=\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}A_n \sin\left(\frac{n\pi x}{L_x}\right)\sin\left(\frac{j\pi y}{L_y}\right)\exp\left[-\left(\frac{k_x n^2\pi^2}{L_x^2}+\frac{k_y j^2\pi^2}{L_y^2}\right)t\right] \quad (17)$$

Where

$$A_n=\frac{120}{nj\pi^2}\left[(-1)^n-1\right]\left[(-1)^j-1\right] \quad (18)$$

E. Error Analysis

Both methods are compared and evaluated based on percentage of relative error calculated as follows (Hoffmann& Chiang, 2000):

$$\text{Relative error (\%)} = \left|\left(\frac{\text{ADI}_{\text{value}}\text{-Analytical}_{\text{value}}}{\text{Analytical}_{\text{value}}}\right)\right|\times 100 \quad (19)$$

III. RESULTS AND DISCUSSION

The temperature distribution is depicted in Figure 3. After 1.2 hrs, the temperature at the middle of the square domain is estimated to reach 1.8266°F. The comparison between the ADI method with the semi-analytical solution is illustrated in Figure 4 and the error analysis is depicted numerically in Table 1 and Table 2 respectively.

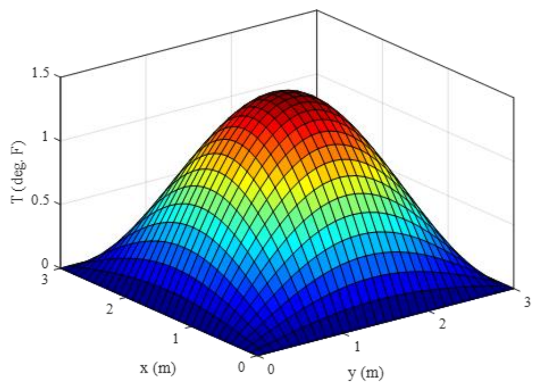

Figure 3. Surface plots of temperature distribution using ADI method

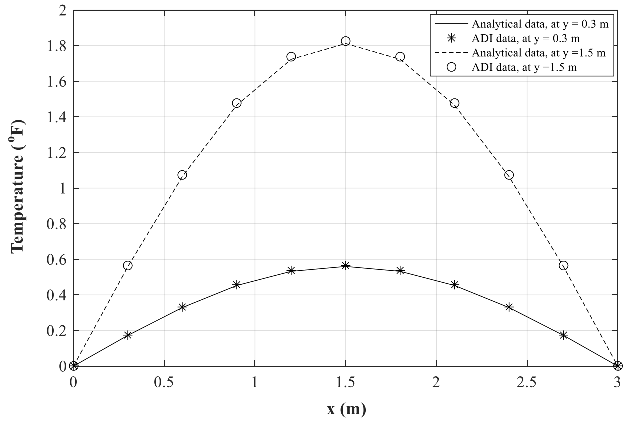

Figure 4. Comparison between ADI method with semi-analytical method for the heat conduction problem along x-axis at $y = 0.3$ m and $y = 1.5$ m for 1.2 hr

Table 1. Relative Error percentage (%) at y=0.3 along *x*-direction

x	T_{ADI}	$T_{analytical}$	Relative error percentage (%)
0.300	0.175	0.173	0.578
0.600	0.332	0.329	0.912
0.900	0.457	0.453	0.883
1.200	0.537	0.533	0.750
1.500	0.565	0.560	0.893
1.800	0.537	0.533	0.750
2.100	0.457	0.453	0.883
2.400	0.332	0.329	0.912
2.700	0.174	0.173	0.578

Table 2. Relative Error percentage (%) at y=1.5 along *y*-direction

x	T_{ADI}	$T_{analytical}$	Relative error percentage (%)
0.300	0.565	0.560	0.893
0.600	1.074	1.065	0.845
0.900	1.478	1.466	0.819
1.200	1.737	1.723	0.813
1.500	1.827	1.812	0.828
1.800	1.737	1.723	0.813
2.100	1.478	1.466	0.819
2.400	1.074	1.065	0.845
2.700	0.564	0.560	0.893

IV. SUMMARY

In overall, the ADI method successfully predicted the temperature distribution with Dirichlet boundary condition with the percentage of error less than 1%. The same method can also be applied to predict the 2-D transient mass diffusion problem particularly the kinetics of percentage carbon penetration during carburising heat treatment.

V. ACKNOWLEDGEMENT

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