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# Analytical Approach to Mixed Integro-Differential Equations of Fractional Order Using Power Series Expansion Principle

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Accurate modeling of many natural phenomena utilizing fractional differential equations is essential to understand the structure, behavior, and construction of these problems. In this article, an analytic-numeric solution of mixed integro-differential equation of fractional-order is presented by using a residual power series expansion principle. This approach constructs to express the solutions in convergent series expansion form with effectively compatible components. Some basic properties for the RPS method are investigated. The numerical example is tested to illustrate the theoretical statements. Numerical results obtained indicate that the exact solution in good agreement with approximate solutions. The main features of the proposed method lie in that it can be directly applied for solving nonlinear fractional problems without the need for unphysical restrictive assumptions, such as linearization, perturbation, or guessing the initial data.

**Keywords:** mixed integro-differential equation; caputo fractional concept; power series expansion

## I. INTRODUCTION

The subject of integro-differential equations (IDEs) of fractional order has received a great deal of interest during the last decades due to their broad applications in the study of complex systems arising in several fields of applied mathematics, physics and engineering. Indeed, the term "fractional calculus" is not new. It is a generalization of classical calculus that deals with the ordinary differentiation and integration of an arbitrary order. Unlike the classical calculus, which has unique concepts and precise physical and geometrical explanations, there are different definitions and concepts of the operations of fractional differentiation and integration as well. Riemann-Liouville, Conformable, Grünwald-Letnikov, Atangana-Baleanu and Caputo are some examples of these definitions (Oldham, K. and Spanier, 1974; Abu Arqub & Al-Smadi, 2018; Moaddy, et al., 2018; Al-Smadi, 2018; Al-Smadi, et al., 2017). The exact solution of such equations is not available in most cases. So, different numerical or analytical techniques have been applied by numerous experts to investigate the approximate solutions for IDEs of fractional order, such as Adomain decomposition method (ADM) (Aladhab, 2016), Homotopy perturbation method (HPM) (Zhang, et al., 2011),Variational iteration method (VIM) (Sweilam, 2007),Fractional differential transform method (DTM) (Arikoglu & Ozkol, 2009),and reproducing kernel method (RKM) (Abu Arqub & Al-Smadi, 2018; Al-Smadi & Arqub, 2019; Abu Arqub, et al., 2018).

The basic aim of this study is to provide the approximate solution of fractional IDEs of Fredholm-Volterra type by using the fractional power series (FPS) method. This method yields Taylor's series expansion of the solutions, as a result, in this case, the exact solutions are available when

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the solutions are polynomials(Abu-Gdairi, et al., 2015; Freihet, et al., 2019; Moaddy, et al., 2015; Al-Smadi, 2019; Komashynska, et al., 2016).Following the RPS procedure, modifications or linearization are not needed when switching from the lowest order to the top. While the proposed method can be applied directly by choosing appropriate values for the initial guessing estimates, then minimizing the residual error terms to reduce computational requirements and to obtain optimal approximation with less time, effort and cost (Momani, S., Arqub, O.A., Freihat, A. and Al-Smadi, M., 2016; Komashynska, et al., 2016; Altawallbeh, et al., 2018).

This work is arranged as follows. In Section II, essential definitions and basic results about the Caputo fractional concept and fractional power series representations are given. The analysis of FPS scheme is presented in Section III. In Section IV, we present one numerical example to show potentiality, generality, and superiority of the method. The last section is dedicated to the conclusion.

## II. BASIC CONCEPTS

In this section, some fundamental definitions and preliminaries about the fractional calculus theory(Abu Arqub & Al-Smadi, 2014; Moaddy, et al., 2017; Hasan, et al., 2019; Podlubny, 1999; El-Ajou, et al., 2015)and fractional power series representations are given.

**Definition 1.** The Riemann-Liouville fractional integral operator of order α is given as

$J^{\alpha}w(t)=\frac{1}{\Gamma(\alpha)}\int_0^t w(\xi)\,(t-\xi)^{1-\alpha}d\xi\,, 0\le\xi<t, \alpha>0.$

For$\alpha=0$then $J^{\alpha}w(t)=w(t)$.

Further, the Riemann-Liouville fractional integral operator has the following:

- $J^{\alpha}J^{\beta}w(t)=J^{\beta}J^{\alpha}w(t)$,
- $J^{\alpha}J^{\beta}w(t)=J^{\alpha+\beta}w(t)$,
- $J^{\alpha}t^{r}=\frac{\Gamma(r+1)}{\Gamma(r+1+\alpha)}t^{r+\alpha}, r>-1.$

**Definition 2.** (Moaddy, et al., 2017) The Caputo fractional derivative of order $\alpha$ is given as:

$$D^{\alpha}w(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{w^{(n)}(\xi)}{(t-\xi)^{\alpha-n+1}}\,d\xi,\ \ n-1<\alpha\le n,$$

On other hand, the operator $D^{\alpha}$ has the following:

- $D^{\alpha}c=0$, for any constant.
- $J^{\alpha}t^{r}=\frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)}t^{r-\alpha}, r>-1.$
- $D^{\alpha}J^{\alpha}w(t)=w(t)$.
- $J^{\alpha}D^{\alpha}w(t)=w(t)-\sum_{j=0}^{n-1}w^{(j)}(\xi^{+})\frac{(t-\xi)^{j\alpha}}{\Gamma(j+1)}$.

**Definition 3**. (Podlubny, 1999) The fractional power series (FPS) about $t=t_0$ represented by

$\sum_{j=0}^{\infty}w_j(t-t_0)^{j\alpha}=w_0+w_1(t-t_0)^{\alpha}+w_2(t-t_0)^{2\alpha}+\cdots,$

where $0\le n-1<\alpha\le n, t\ge t_0,$ and $w_j$'s are constants coefficients of the series.

**Theorem 1.** (El-Ajou, et al., 2015) Suppose that $h(t)$ has the FPS representation at $t=t_0$as follows

$h(t)=\sum_{j=0}^{\infty}w_m(t-t_0)^{j\alpha}\,,\ t_0\le t\le t_0+a.$

If $h(t)\in C[t_0,t_0+a)$, and $D^{j\alpha}h(t)\in C(t_0,t_0+a)$, for $j=0,1,2,\ldots$, then thecoefficients $w_j$ willtake the form $w_j=\frac{D^{j\alpha}h(t_0)}{\Gamma(j\alpha+1)}$, where $D^{j\alpha}=D^{\alpha}\cdot D^{\alpha}\cdots D^{\alpha}$ ($j$-times).

## III. ANALYSIS OF FRACTIONAL POWER SERIES ALGORITHM

In order to illustrate the basic procedure of the FPS technique, the following fractional Fredholm–Volterraintegro-differential equation are considered

$$D^{\alpha}w(t)+\int_a^b k(t,\xi)w(\xi)\,d\xi+\int_a^t h(t,\xi)\,w(\xi)d\xi=f(t), \tag{1}$$

subject to the initial condition

$$w(0)=w_0. \tag{2}$$

where$\in(0,1]$, $a\le t,\xi\le b, f:[a,b]\to\mathbb{R}$, is acontinuous real-valued function and $k(t,\xi)$, $h(t,\xi)$ are two continuous arbitrary kernel functions, while $D^{\alpha}$ stands to Caputo fractional derivative.

Regarding applying the FPS method (Momani, S., Arqub, O.A., Freihat, A. and Al-Smadi, M., 2016; Komashynska, et al., 2016; Altawallbeh, et al., 2018), the solution of Eqs.(1)and (2) can be expressed as FPS expansion about t = 0 of the form

$$w(t)=\sum_{n=0}^{\infty}w_n\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{3}$$

where $w(0)=w_0$, so the series solution (3) will be as

$$w(t)=w_0+\sum_{n=1}^{\infty}w_n\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{4}$$

We can approximate the series solution (4), by the $k$-th truncated series

$$w_k(t) = w_0 + \sum_{n=1}^{k} w_n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \quad (5)$$

According to the FPS method, we define the $k$-th residual function $Res_k(t)$ for Eqs.(1)and (2) as

$$Res_k(t) = D^{\alpha} w_k(t) - \int_a^b k(t,\xi) w_k(\xi)\, d\xi - \int_a^t h(t,\xi)\, w_k(\xi) d\xi - f(t). \quad (6)$$

Further, we define the following residual function $Res(\mathrm{t})$ as follows

$$Res(\mathrm{t}) = \lim_{k\to\infty} Res_k(t) = D^{\alpha} w_k(t) - \int_a^b k(t,\xi) w_k(\xi)\, d\xi - \int_a^t h(t,\xi)\, w_k(\xi) d\xi - f(t). \quad (7)$$

Here, we noted that $Res(\mathrm{t}) = 0$ for all $\geq 0$ . Also, $D^{\alpha} Res(\mathrm{t}) = 0$. Moreover, $D^{m\alpha}\mathrm{Res_q}(0) = D^{m\alpha} Res_k(0) = 0$ for$m = 1,2,\dots,k$.

Consequently, the following differential equation of fractional order assist us to determine the value of the coefficients$w_n$, for $n = 1,2,\dots,k$

$$D^{(k-1)\alpha} Res_k(0) = 0,\ k = 1,2,3,\dots. \quad (8)$$

In view of that to obtain the unknown coefficients $w_n$, for $n = 1,2,\dots,k$ of Eq. (3), write the $k$-th truncated series into the $k$-th residual Eq. (6), find$D^{(k-1)\alpha} Res_k(t)$for$k = 1,2,3,\dots$, substitute$t = 0$ in the resulting equation and then equal it by zero.

## IV. NUMERICAL EXAMPLE

The RPSM is practical as well as useful to solve not only differential equations but also the integral and integro-differential equations. This section is concerned with applying the proposed method to demonstrate the simplicity and effectiveness for solving mixed IDEs of fractional order. The method is implemented directly with no required to transformation or restrictive assumptions. Numeric outcomes indicate that the present approach is very convenient for solving such problems. Anyhow, we all know that the algorithms have a limited set of principles for performing calculations on the computer with specific digits so that principles are determined at each instant exactly what the computer must do afterward.

Consider the mixed IDE in the following form:

$$D^{\alpha} w(t) + \int_0^1 \sin(t) w(\xi)\, d\xi - \frac{1}{2}\int_0^t \xi\, w(\xi) d\xi = 1 - \frac{1}{2} t(t-4)e^t + sin(t), \quad (9)$$

subject to initial condition

$$w(t) = 0. \quad (10)$$

The exact solution at $\alpha = 1$ is given by $w(t) = te^t$.

This example explores more large scale to apply the RPS algorithm for solving Eqs. (9) and (10). To do so, we construct appropriate residual functions, and we simplify the used the RPS algorithm and computations step by step.

The $k$-th residual function $Res_k(t)$ is given by

$$Res_k(t) = D^{\alpha} w_k(t) + \int_0^1 \sin(t) w_k(\xi)\, d\xi - \frac{1}{2}\int_0^t \xi\, w_k(\xi) d\xi - 1 + \frac{1}{2} t(t-4)e^t - \sin(t), \quad (11)$$

where$w_k(t)$ has the form

$$w_k(t) = \sum_{n=1}^{k} w_n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

The exact and approximate solutions are compared in Table 1. The results obtained by the RPS method show that the exact solutions are in good agreement with approximate solutions at $\alpha = 1$, $n = 10$ and step size 0.16.

Table 1. Results of solutions at $\alpha = 1$

| $t$ | Exact | RPS | Abs. Error | Rel. Error |
|---|---|---|---|---|
| 0.16 | 0.1877 | 0.1877 | $4.9 \times 10^{-16}$ | $2.6 \times 10^{-15}$ |
| 0.32 | 0.4406 | 0.4406 | $1.0 \times 10^{-12}$ | $2.3 \times 10^{-12}$ |
| 0.48 | 0.7757 | 0.7757 | $9.0 \times 10^{-11}$ | $1.2 \times 10^{-10}$ |
| 0.64 | 1.2137 | 1.2137 | $2.2 \times 10^{-9}$ | $1.8 \times 10^{-9}$ |
| 0.80 | 1.7804 | 1.7804 | $3.0 \times 10^{-8}$ | $1.4 \times 10^{-8}$ |
| 0.96 | 2.5072 | 2.5072 | $1.9 \times 10^{-7}$ | $7.7 \times 10^{-8}$ |

As consequence, the solutions obtained are smooth and convergent to the approximate ones as well as the capability of the process to handle different interesting numerical examples where no time discretization is considered for computations.

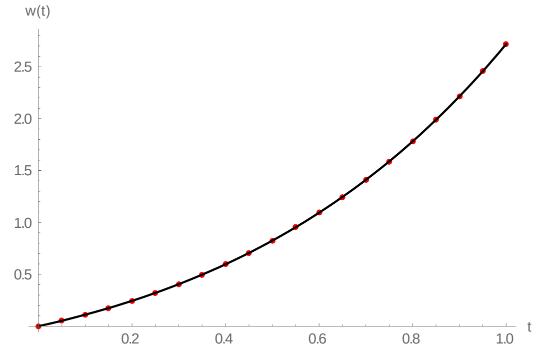


Figure 1. Solution plots of the exact and approximation at $\alpha = 1$ (---- Exact, *** approximate)

## V. SUMMARY

In this article, the fractional power series (FPS) algorithm has been applied successfully for providing RPS approximate solution of fractional integro-differential equations of Fredholm-Volterra type. This technique based on the residual error functions and generalized Taylor series to derive the FPS solution without linearization, perturbation, or discretization. The results indicate that the present method is extremely effective for solving such of these IDEs. Thus, the FPS technique is powerful and convenient to derive the approximate solutions for such these of fractional differential equations.

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