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Trigonometrically-Fitted Diagonally Implicit Two Derivative Runge-Kutta method for the Numerical Solution of Periodical IVPs

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A Trigonometrically-Fitted Diagonally Implicit Two Derivative Runge-Kutta (TFDITDRK) method for the numerical solution of first order Initial Value Problems (IVPs) which possesses oscillatory solutions is derived. We present a fourth-order two stage Diagonally Implicit Two Derivative Runge-Kutta (DITDRK) method designed using the trigonometrically-fitted property. The numerical experiments are carried out to show the efficiency of the derived methods in comparison with other existing Runge-Kutta (RK) methods of the same order and properties. **Keywords:** Diagonally Implicit methods, IVPs, ODEs, TDRK methods, Trigonometrically-Fitted.

I. Introduction

Consider the numerical solution of the Initial Value Problems (IVPs) for first order Ordinary Differential Equations (ODEs) in the form of

$$y^{'} = f(x, y), \qquad y(x_0) = y_0, \qquad (1)$$

whose solutions show an observable periodically or oscillatory behavior. Such problems occurs in several fields of applied sciences, for instance, circuit simulation, orbital mechanics, molecular dynamics, mechanics, astrophysics and electronics which has attracted the concern of numerous researchers. In general, most problems with oscillatory or periodically behavior are second order or higher order. Hence, it is quite important to reduce the higher order problems to first order problems in order to solve the ODEs (1).

Vanden et al. (1999) and Berghe et al. (1999) constructed exponentially-fitted explicit RK methods which exactly integrates differential IVPs problems whose solutions are linear combination of the functions in the form of $exp(\omega x)$ and $exp(-\omega x)$. Meanwhile, Simos (1998) constructed exponentially-fitted and trigonometrically-fitted RK methods of fourth-order for the numerical integration of IVPs with periodic solutions.

Several well-known authors in their papers have developed Diagonally Implicit Runge-Kutta (DIRK) methods especially designed for solving stiff problems. Franco and Gómez (2003) focused on dispersion (phase errors) as well as the dispersion conditions for symmetric DIRK methods and symmetric stability functionsand succeeded in developing two new fourth-order symmetric methods of four and five stages. Skvortsov (2006) developed methods of third, fourth, fifth and sixth order which have several advantages over some other methods in terms of minimization of certain error functions.

Ababneh et al. (2009) presented new fifth-order DIRK integration formulas for stiff IVPs which are designed to be L-stable method. The stability of the method derived is analyzed and numerical results are carried out to verify the conclusions. Ismail et al. (2009) and Jawias et al. (2010) developed SDIRK methods for solving linear ODEs of fifth-order five stage and fourth-order four stage respectively where the numerical results have proven that both methods have bigger stability region compared to explicit methods.

In the evolution of TDRK methods, Chan and Tsai (2010) introduced special explicit TDRK methods by implementing and including the second derivative which involves only one evaluation of f and a few evaluations of g per step. They managed to derive TDRK methods of seventh-order and some embedded pairs. Zhang et al. (2013) developed a new trigonometrically-fitted TDRK method of algebraic order five for solving Schrödinger equation and related problems. The linear stability as well as the phase properties of the new method are analyzed.

Tsai et al. (2014) presented their study on both explicit and implicit TDRK methods on stiff ODEs problems and extend their work by implementing the developed methods to various Partial Differential Equations (PDEs). Chen et al. (2015) constructed three practical exponentially fitted TDRK (EFTDRK) methods for the simulation of oscillatory genetic regulatory systems. Yakubu and Kwami (2015) introduced a new class of implicit TDRK collocation methods especially for the numerical solution of systems of equations and their implementation in an efficient parallel computing environment.

In this recent year, there are no research findings related to trigonometrically-fitting in DITDRK methods. Researchers have not yet explored the advantages or disadvantages of applying trigonometrically fitted techniques to DITDRK methods. Hence, in this paper, fourth-order two stage trigonometrically-fitted DITDRK method is derived. In Section 2, an overview of TDRK method is given. In Section 3, trigonometrically-fitted conditions are considered. The construction of the trigonometrically-fitted DITDRK method is given in Section 4. The numerical results, discussion, and conclusion are discussed briefly in Sections 5, 6, and 7, respectively.

II. Diagonally Implicit Two Derivative Runge-Kutta method

A TDRK method for the numerical integration of IVPs (1) is given by

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + h^2 \sum_{j=1}^{s} \hat{a}_{ij} g(Y_j), \tag{2}$$

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(Y_i) + h^2 \sum_{i=1}^{s} \hat{b}_i g(Y_i), \tag{3}$$

where $i = 1, \ldots, s$.

The TDRK parameters $a_{ij}, \hat{a}_{ij}, b_i, \hat{b}_i$ and c_i are assumed to be real and s is the number of stages of the method. The s-dimensional vectors $b, \hat{b}, c$ and $s \times s$ matrix, A and $\hat{A}$ are introduced where $b = [b_1, b_2, \ldots, b_s]^T$, $\hat{b} = \left[\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_s\right]^T$, $c = [c_1, c_2, \ldots, c_s]^T$, $A = [a_{ij}]$ and $\hat{A} = [\hat{a}_{ij}]$ respectively.

The TDRK method with the coefficients in (2) and (3) are presented using the Butcher tableau as follows:

$$\begin{array}{c|c||c} c & A & \hat{A} \\ \hline & b^T & \hat{b}^T \end{array}$$

Diagonally implicit methods with a minimal number of function evaluations can be developed by considering the methods in the form

$$Y_i = y_n + hc_i f(x_n, y_n) + h^2 \sum_{j=1}^{i} \hat{a}_{ij} g(x_n + hc_j, Y_j), \quad (4)$$

$$y_{n+1} = y_n + hf(x_n, y_n) + h^2 \sum_{i=1}^{s} \hat{b}_i g(x_n + hc_i, Y_i), \quad (5)$$

where $i = 1, \ldots, s$.

The above method is denoted as special DITDRK method. The unique part of this method is that it involves only one evaluation of f and many evaluation of g per step compared to many evaluation of f per step in traditional RK methods. Its Butcher tableau is given as follows:

c	$\hat{A}$
	$\hat{b}^T$

The order conditions for special DITDRK methods as written in Chan and Tsai (2010) are given in Table 1.

Table 1: Order conditions for special DITDRK methods.

Order	Conditions	
1	$b^T e = 1$	
2	$\hat{b}^T e = \frac{1}{2}$	
3	$\hat{b}^T c = \frac{1}{6}$	
4	$\hat{b}^T c^2 = \frac{1}{12}$	
5	$\hat{b}^T c^3 = \frac{1}{20}$	$\hat{b}^T \hat{A} c = \frac{1}{120}$
6	$\hat{b}^T c^4 = \frac{1}{30}$	$\hat{b}^T c\hat{A} c = \frac{1}{180}$
	$\hat{b}^T \hat{A} c^2 = \frac{1}{360}$	
7	$\hat{b}^T c^5 = \frac{1}{42}$	$\hat{b}^T c^2 \hat{A} c = \frac{1}{252}$
	$\hat{b}^T c\hat{A} c^2 = \frac{1}{504}$	$\hat{b}^T \hat{A} c^3 = \frac{1}{840}$
	$\hat{b}^T \hat{A}^2 c = \frac{1}{5040}$	

III. Trigonometrically-Fitted Property

In order to construct the trigonometrically-fitted method, we introduce an extra parameter z_i in the internal stage given by equation (4) of the DITDRK method,

$$Y_i = y_n + hz_i c_i f + h^2 \sum_{j=1}^{i} \hat{a}_{ij} g(x_n + hc_j, Y_j), \quad (6)$$

$$y_{n+1} = y_n + hf + h^2 \sum_{i=1}^{s} \hat{b}_i g(x_n + hc_i, Y_i), \quad (7)$$

where $i = 1, \ldots, s$.

Meanwhile, the associated Bucther tableau has an extra column

c	z	$\hat{A}$
		$\hat{b}^T$

A DITDRK method (6)–(7) will have trigonometrically-fitted properties if it exactly integrate the functions $e^{i\lambda x}$ and $e^{-i\lambda x}$ or equivalently $\sin(\lambda x)$ and $\cos(\lambda x)$ with the principal frequency of the problem, $\lambda > 0$ when it is applied to the test equation $y' = i\lambda y$ and $y'' = -\lambda^2 y$. When DITDRK method (6)–(7) is applied to the test question as stated above, the method becomes:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{s} \hat{b}_i g(x_n + c_i h, Y_i), \quad (8)$$

where

$$Y_1 = y_n + z_1 c_1 h y'_n - h^2 \lambda^2 \hat{a}_{11} Y_1, \quad (9)$$

$$Y_2 = y_n + z_2 c_2 h y'_n - h^2 \lambda^2 \left(\hat{a}_{21} Y_1 + \hat{a}_{22} Y_2\right), \quad (10)$$

$$Y_3 = y_n + z_3 c_3 h y'_n - h^2 \lambda^2 \left(\hat{a}_{31} Y_1 + \hat{a}_{32} Y_2 + \hat{a}_{33} Y_3\right), \quad (11)$$

$$\vdots$$

$$Y_i = y_n + z_i c_i h y'_n - h^2 \lambda^2 \sum_{j=1}^{i} \hat{a}_{ij} (Y_j). \quad (12)$$

which results in

$$y_{n+1} = y_n + hy'_n - h^2\lambda^2 \sum_{i=1}^{s} \hat{b}_i(Y_i). \quad (13)$$

Let $y_n = e^{i\lambda x}$ and $f(x_n, y_n) = i\lambda y_n$, compute the values for y_{n+1} and substitute those values in equation (8)-(13). By using $e^{iv} = \cos(v)+i\sin(v)$, separate the real and the imaginary part, hence the following equations are obtained

$$\cos(c_i v) - 1 + v^2 \sum_{j=1}^{i} \hat{a}_{ij} \cos(c_j v) = 0, \quad (14)$$

$$\sin(c_i v) - z_i c_i v + v^2 \sum_{j=1}^{i} \hat{a}_{ij} \sin(c_j v) = 0, \quad (15)$$

together with

$$\cos(v) - 1 + v^2 \sum_{i=1}^{s} \hat{b}_i \cos(c_i v) = 0, \quad (16)$$

$$\sin(v) - v + v^2 \sum_{i=1}^{s} \hat{b}_i \sin(c_i v) = 0. \quad (17)$$

where $i = 1, \ldots, s$ and $v = \lambda h$.

IV. Derivation of TFDITDRK method

Firstly, we will derive the fourth-order two stages DITDRK method. According to the order conditions up to order four in Table 1, we have

$$\hat{b}_2 + \hat{b}_3 - \frac{1}{2} = 0, \quad (18)$$

$$\hat{b}_2 c_2 + \hat{b}_3 c_3 - \frac{1}{6} = 0, \quad (19)$$

$$\hat{b}_2 {c_2}^2 + \hat{b}_3 {c_3}^2 - \frac{1}{12} = 0. \quad (20)$$

Solving equation (18)-(20) we obtain $\hat{b}_1, \hat{b}_2$ and c_2 in term of c_1

$$\hat{b}_1 = \frac{1}{(36\,{c_1}^2 - 24\,c_1 + 6)}, \quad (21)$$

$$\hat{b}_2 = \frac{1}{3}\left(\frac{9\,{c_1}^2 - 6\,c_1 + 1}{6\,{c_1}^2 - 4\,c_1 + 1}\right), \quad (22)$$

$$c_2 = \frac{1}{2}\left(\frac{2\,c_1 - 1}{3\,c_1 - 1}\right). \quad (23)$$

Our aim is to choose c_1 such that the principal local truncation error coefficient, $\left\|\tau^{(5)}\right\|_2$ have a very small value. Wrong choices of c_1 may cause a huge global error difference. By plotting the graph of $\left\|\tau^{(5)}\right\|_2$ against c_1, a small value of c_1 is chosen in the range of $[0.0, 1.0]$ and hence, the value of c_1 lies between $[0.1, 0.3]$. We choose $c_1 = \frac{1}{5}$ for an optimized pair. All the coefficients are showed in the following Butcher tableau and it is denoted as DITDRK(2,4).

Table 2: Butcher Tableau for DITDRK(2,4) Method

$\frac{1}{5}$	$\frac{1}{50}$	
$\frac{3}{4}$	$\frac{209}{800}$	$\frac{1}{50}$
	$\frac{25}{66}$	$\frac{4}{33}$

The norms of the principal local truncation error coefficients for DITDRK(2,4) method is given by

$$\left\|\tau^{(5)}\right\|_2 = 4.374801584 \times 10^{-3}. \quad (24)$$

Now, the trigonometrically-fitted property will be implemented in the method derived earlier. Evaluate the internal stage given by equation (14)–(15) and retaining the value of c_1 and c_2 in DITDRK(2,4), we have

$$-1 + \hat{a}_{11} v^2 \cos\left(\frac{v}{5}\right) + \cos\left(\frac{v}{5}\right) = 0, \quad (25)$$

$$-1 + v^2\left(\hat{a}_{2,1}\cos\left(\frac{v}{5}\right) + \hat{a}_{1,1}\cos\left(\frac{3}{4}v\right)\right) + \cos\left(\frac{3}{4}v\right) = 0, \quad (26)$$

$$-\frac{1}{5}\,z_1 v + \hat{a}_{1,1} v^2 \sin\left(\frac{v}{5}\right) + \sin\left(\frac{v}{5}\right) = 0, \quad (27)$$

$$-\frac{3}{4}\,z_2 v + v^2\left(\hat{a}_{2,1}\sin\left(\frac{v}{5}\right) + \hat{a}_{1,1}\sin\left(\frac{3}{4}v\right)\right) + \sin\left(\frac{3}{4}v\right) = 0, \quad (28)$$

together with the final stage given by equation (16)–(17)

$$-1+v^2\left(\hat{b}_1\cos\left(\frac{v}{5}\right)+\hat{b}_2\cos\left(\frac{3}{4}v\right)\right)+\cos(v)=0, \quad (29)$$

$$-v+v^2\left(\hat{b}_1\sin\left(\frac{v}{5}\right)+\hat{b}_2\sin\left(\frac{3}{4}v\right)\right)+\sin(v)=0. \quad (30)$$

Solve equations (25)–(30) lead to

$$\hat{a}_{11}=\frac{-\cos\left(\frac{v}{5}\right)+1}{\cos\left(\frac{v}{5}\right)v^2}, \quad (31)$$

$$\hat{a}_{21}=\frac{2\cos\left(\frac{v}{5}\right)-2\cos\left(\frac{3}{4}v\right)}{v^2\cos\left(\frac{2}{5}v\right)+v^2}, \quad (32)$$

$$\hat{b}_1=\frac{-\cos\left(\frac{3}{4}v\right)v+\sin\left(\frac{v}{4}\right)+\sin\left(\frac{3}{4}v\right)}{v^2\sin\left(\frac{11}{20}v\right)}, \quad (33)$$

$$\hat{b}_2=\frac{-\sin\left(\frac{4}{5}v\right)+v\cos\left(\frac{v}{5}\right)-\sin\left(\frac{v}{5}\right)}{v^2\sin\left(\frac{11}{20}v\right)}, \quad (34)$$

$$z_1=5\,\frac{\sin\left(\frac{v}{5}\right)}{v\cos\left(\frac{v}{5}\right)}, \quad (35)$$

$$z_2=\frac{8\sin\left(\frac{11}{20}v\right)+4\sin\left(\frac{2}{5}v\right)}{3v\cos\left(\frac{2}{5}v\right)+3v}. \quad (36)$$

As $v\to 0$, the following Taylor expansions are obtained

$$\hat{a}_{11}=\frac{1}{50}+\frac{v^2}{3000}+\frac{61v^4}{11250000}+\frac{277v^6}{3150000000}+\frac{50521v^8}{35437500000000}+\dots, \quad (37)$$

$$\hat{a}_{21}=\frac{209}{800}-\frac{10241v^2}{3840000}+\frac{50369v^4}{46080000000}-\frac{282511361v^6}{1032192000000000}-\frac{30372761353v^8}{5308416000000000000}+\dots, \quad (38)$$

$$\hat{b}_1=\frac{25}{66}-\frac{v^2}{792}+\frac{811v^4}{16632000}+\frac{907v^6}{665280000}+\frac{55546409v^8}{1317254400000000}+\dots, \quad (39)$$

$$\hat{b}_2=\frac{4}{33}+\frac{v^2}{792}+\frac{6217v^4}{133056000}+\frac{147557v^6}{106444800000}+\frac{7133716823v^8}{168608563200000000}+\dots, \quad (40)$$

$$z_1=1+\frac{v^2}{75}+\frac{2v^4}{9375}+\frac{17v^6}{4921875}+\frac{62v^8}{1107421875}+\dots, \quad (41)$$

$$z_2=1-\frac{49v^2}{12000}-\frac{1547v^4}{19200000}-\frac{1314443v^6}{537600000000}-\frac{75866893v^8}{1327104000000000}+\dots. \quad (42)$$

This new method is denoted as TFDITDRK(2,4). TFDITDRK(2,4) method will reduce to its original method which is denoted as DITDRK(2,4) method when $v\to 0$. Other than that, as $v\to 0$, TFDITDRK(2,4) method will have the same error constant as DITDRK(2,4) method.

V. Problems Tested and Numerical Results

In this section, the performance of the proposed method TFDITDRK(2,4) are compared with existing RK methods with the trigonometrically-fitted, phase-fitted and amplification-fitted properties by considering the following problems. All problems below are tested using C code for solving differential equations where the solutions are periodic.

Problem 1(Harmonic Oscillator)

$$y_1{}'(x)=y_2(x), \qquad y_1(0)=1.0,$$
$$y_2{}'(x)=-64y_1(x), \qquad y_2(0)=-2.0,$$

for $x\in[0,1000]$. Exact solution is

$$y_1(x)=-\frac{1}{4}\sin(8x)+\cos(8x),$$
$$y_2(x)=-2\cos(8x)-8\sin(8x).$$

Problem 2(Inhomogeneous problem, (Van de Vyver, 2007))

$$y_1{}'=y_2, \qquad y_1(0)=1,$$
$$y_2{}'=-100y_1+99\sin(x), \qquad y_2(0)=11,$$

for $x\in[0,1000]$. Exact solution is

$$y_1(x)=\cos(10x)+\sin(10x)+\sin(x),$$
$$y_2(x)=-10\sin(10x)+10\cos(10x)+\cos(x).$$

Problem 3(An "almost" Periodic Orbit problem, (Stiefel and Bettis, 1969))

$$\begin{aligned} y_1{}' &= y_2, & y_1(0) &= 1, \\ y_2{}' &= -y_1 + 0.001\cos(x), & y_2(0) &= 1, \\ y_3{}' &= y_4, & y_3(0) &= 0, \\ y_4{}' &= -y_3 + 0.001\sin(x), & y_4(0) &= 0.995, \end{aligned}$$

for $x \in [0, 1000]$. Exact solution is

$$\begin{aligned} y_1(x) &= \cos(x) + 0.0005x\sin(x), \\ y_2(x) &= -\sin(x) + 0.0005x\cos(x) + 0.0005x\sin(x), \\ y_3(x) &= \sin(x) - 0.0005x\cos(x), \\ y_4(x) &= \cos(x) + 0.0005x\sin(x) - 0.0005\cos(x). \end{aligned}$$

Problem 4 (Duffing problem, (Kosti et al., 2012))

$$\begin{aligned} y_1{}' &= y_2, \\ y_2{}' &= -y_1 - y_1{}^3 + 0.002\cos(1.01x), \\ y_1(0) &= 0.200426728067, \\ y_2(0) &= 0, \end{aligned}$$

for $x \in [0, 100]$. Exact solution is

$$\begin{aligned} y_1(x) =& 0.200179477536\cos(1.01x) + \\ & 2.46946143 \times 10^{-4}\cos(3.03x) + \\ & 3.04014 \times 10^{-7}\cos(5.05x) + \\ & 3.74 \times 10^{-10}\cos(7.07x), \\ y_2(x) =& -0.2021812723\sin(1.01x) - \\ & 7.482468133 \times 10^{-4}\sin(3.03x) - \\ & 1.53527070 \times 10^{-6}\sin(5.05x) - \\ & 2.64418 \times 10^{-9}\sin(7.07x). \end{aligned}$$

Problem 5 (Prothero-Robinson problem, (Chan and Tsai, 2010))

$$y' = -\lambda(y - \varphi) + \varphi', \quad y(0) = \varphi(0), \quad Re(\lambda) < 0,$$

where $\varphi(x)$ is a smooth function and $\varphi(x) = \sin(x)$ for $x \in [0, 1000]$. Exact solution is $y(x) = \varphi(x)$.

The following abbreviations are used in Figures 1–10.

- TFDITDRK(2,4): The fourth-order two stages trigonometrically-fitted DITDRK method derived in this paper.
- TFDIRKK(3,4): Existing fourth-order three stages trigononometrically-fitted DIRK method developed in Kalogiratou (2013).
- PFAFDIRKA(3,4): Existing fourth-order three stages phase-fitted and amplification-fitted DIRK method given by Ahmad et al. (2016).
- EFDIRKE(3,4): Existing fourth-order three stages exponentially-fitted DIRK method given in Ehigie et al. (2018).

The performance of these numerical results are represented graphically in Figures 1–10.

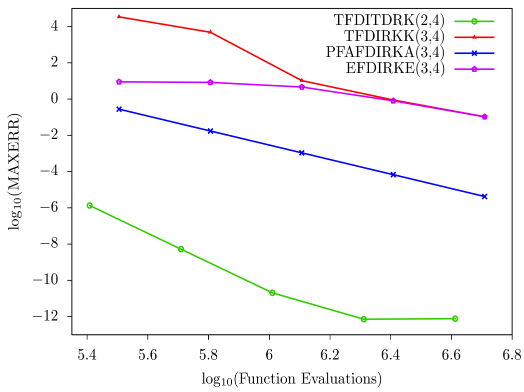

Figure 1: The efficiency curve for Harmonic Oscillator (Problem 1) for TFDITDRK(2,4) method with $h = 1.0/2^i, i = 5, \ldots, 9$.

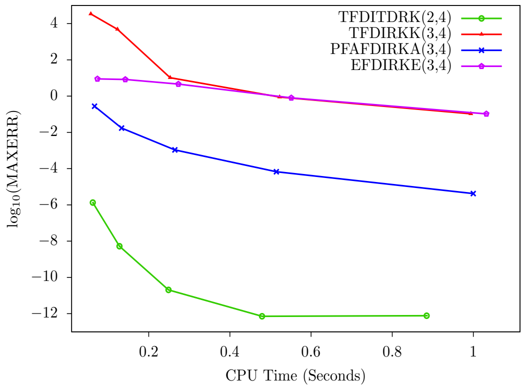

Figure 2: The efficiency curve for Harmonic Oscillator (Problem 1) for TFDITDRK(2,4) method with $h = 1.0/2^i, i = 5, \ldots, 9$.

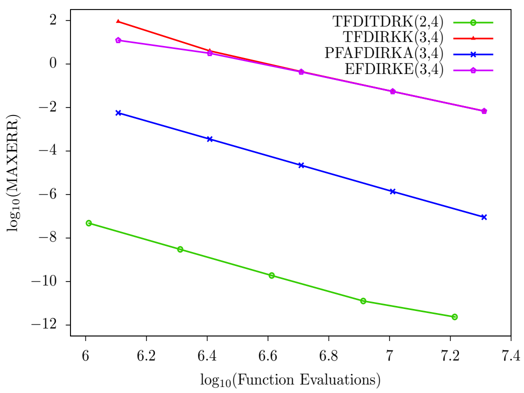

Figure 3: The efficiency curve for Inhomogeneous problem (Problem 2) for TFDITDRK(2,4) method with $h = 1.0/2^i, i = 7, \ldots, 11$.

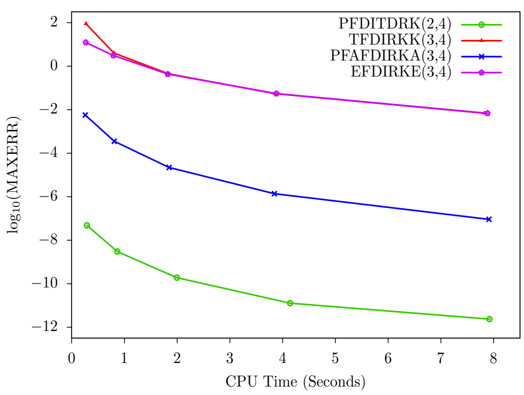

Figure 4: The efficiency curve for Inhomogeneous problem (Problem 2) for TFDITDRK(2,4) method with $h = 1.0/2^i, i = 7, \ldots, 11$.

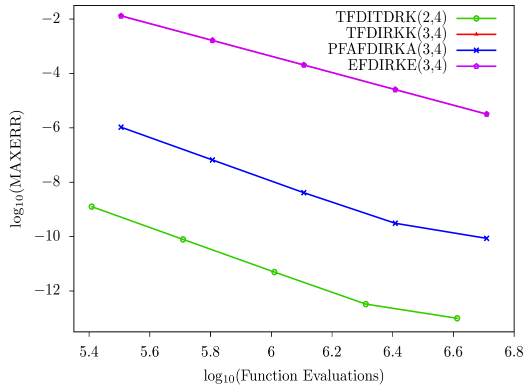

Figure 5: The efficiency curve for an "almost" Periodic Orbit problem (Problem 3) for TFDITDRK(2,4) method with $h = 1.0/2^i, i = 5, \ldots, 9$.

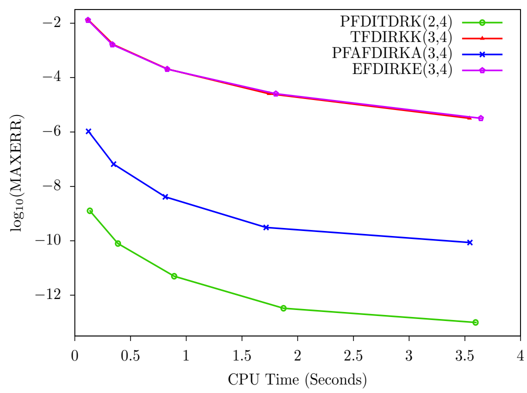

Figure 6: The efficiency curve for an "almost" Periodic Orbit problem (Problem 3) for TFDITDRK(2,4) method with $h = 1.0/2^i, i = 5, \ldots, 9$.

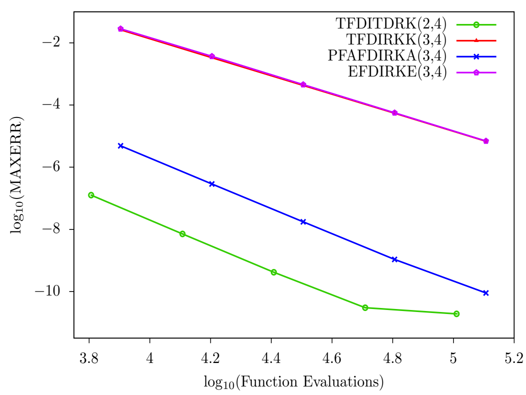

Figure 7: The efficiency curve for Duffing problem (Problem 4) for TFDITDRK(2,4) method with $h = 1.0/2^i, i = 3, \ldots, 7$.

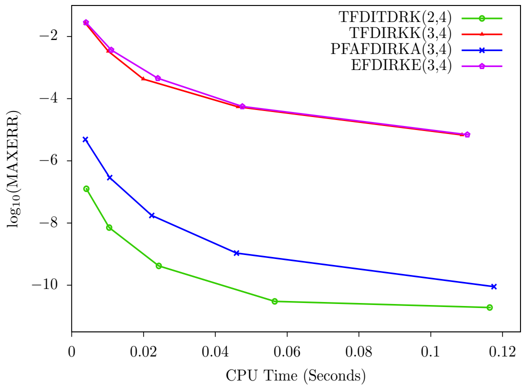

Figure 8: The efficiency curve for Duffing problem (Problem 4) for TFDITDRK(2,4) method with $h = 1.0/2^i, i = 3, \ldots, 7$.

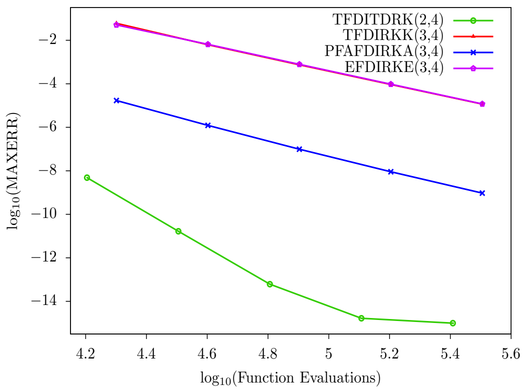

Figure 9: The efficiency curve for Prothero-Robinson problem (Problem 5) for TFDITDRK(2,4) method with $h = 1.0/2^i, i = 1, \ldots, 5$.

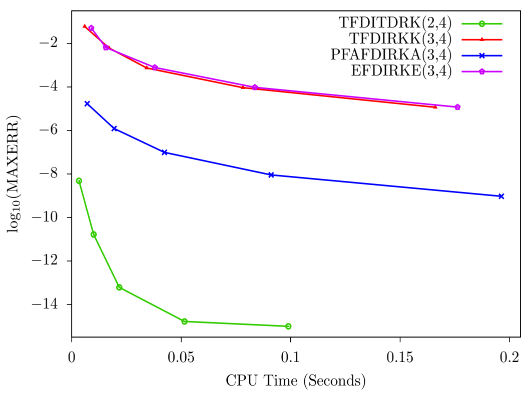

Figure 10: The efficiency curve for Prothero-Robinson problem (Problem 5) for TFDITDRK(2,4) method with $h = 1.0/2^i, i = 1, \ldots, 5$.

VI. Discussion

The results show the typical properties of the trigonometrically-fitted DITDRK method, TFDITDRK(2,4) which have been derived earlier. The derived method are compared with some well-known existing RK methods with the same order and properties. The global error and the efficiency of the method over a long period of integration are plotted. Figures 1–10 represent the efficiency and accuracy of the method developed by plotting the graph of the logarithm of the maximum global error against the logarithm number of function evaluations for a longer periods of computations as well as the CPU times in seconds.

From the plotted graphs, TFDITDRK(2,4) method has the smallest maximum global error compared to other existing RK methods which have trigonometrically-fitted, exponentially-fitted and phase-fitted and amplification-fitted properties. In DITDRK methods, the existance of the second derivative which involves only one evaluation of f and a few evaluations of g per step compared to the traditional DIRK methods which only need the first derivative that makes the DITDRK methods more advanced. Hence, this is why TFDITDRK(2,4) method can achieve smaller maximum global error.

In Figures 2 and 10, TFDITDRK(2,4) have shorter CPU times in comparison with other existing methods. Meanwhile in Figures 4, 6 and 8, TFDITDRK(2,4) have comparable CPU times compared to other existing methods. The comparisons are made between methods of the same properties but it can be seen that TFDITDRK(2,4) method is the most accurate method of all in term of maximum global error and comparable in term of CPU times. TFDITDRK(2,4) method has lesser number of stages to other existing DIRK methods. Hence, fewer number of stages lead to a fewer total number of function evaluations.

VII. Conclusion

In this research, a trigonometrically-fitted DITDRK method of fourth-order is developed. Based on the numerical results obtained, it can be concluded that the derived method, TFDITDRK(2,4) is more promising compared to other well-known existing DIRK methods in terms of accuracy, the number of function evaluations per step and comparable in term of CPU times.

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