ASM Sc. J., 12, Special Issue 1, 2019 for IQRAC2018, 256-264

Adaptive Adjusted Compound Smoother in Recovering Signal from Noise

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Compound smoother is a non-linear smoothing technique that has the ability to remove heavy noise from signal and at the same time is resistant to sudden changes and impulse in the data series. In this study, compound smoother of 4253HT was adjusted in the algorithm specifically to estimate the middle point of running median for even span by applying the following types of means; geometric, harmonic, quadratic and contraharmonic. Simulations were conducted by generating special functions of Doppler, Bumps, Blocks and Heavy Sine with noise that produced a few outliers and high volatility. The regression coefficients show that adaptive 4253HT to perform the best in removing long tailed and heavy noise. Results from estimated integrated mean square error show that adaptive 4253HT managed to extract signals of Doppler, Block and Bumps from 10% contaminated normal error. Adaptive 4253HT also was observed to work best in the recovery of the signal of Bumps from noise with high volatility. Practical application on the daily amount of rainfall which was conducted, asserts that if heavy rain started to occur, it continued on for another four days on average.
Keywords : Compound smoother, adaptive 4253HT, running median, signal, noise

I. Introduction

Most data analysis and practically all of exploratory data analysis consist of the objective to look for patterns in a data. Smoothing is a process of obtaining signal from an irregular sequence of data values. The main purpose of smoothing a time series is to remove noises in a data so that the smoothed values may produce a good signal of possible trend and shape of the distribution.

Linear smoothing technique has dominated for a long time ago due to its simplicity in implementation and analysis, (Gabbouj et al., 1992). It has been recognized that linear smoother is optimal in eliminating Gaussian noise and tracking trends, which is quite common in practice, (Bernholt et al., 2006). Unfortunately, they are highly vulnerable to outliers and cannot deal well with nonlinearity in a data series. Sudden changes in the series, blurs edges, leading to the lost of important information, (Bernholt et al., 2006).

(Tukey, 1977) introduces a running median smoother which is robust to outliers and at the same time has the ability to preserve edges during abrupt changes. Running median has been acknowledged as a powerful statistical tools in extracting signal from possibly long-tailed or occasionally "spikey" noise (Velleman, 1980). Standard median smoother is well known for properties of noise elimination (elimination of noise with heavy-tailed distribution) and edge preservation. However, running median tends to eliminate Gaussian noise, resulting in oversmoothing as important signal is destroyed in a series, (Velleman, 1980). Over the years, running median has been extended to many versions for improvement including repeated running median, weighted running median, recursive running median, windsorized smoother and compound smoother.

(Tukey, 1977) sparked the idea of compound smoother and has then extended to a few different interesting versions including repeating, splitting, Hanning and re-smoothing the rough (Velleman, 1980). The most common and established technique of compound smoother is 4253HT. This method has been highly applicable in removing spikes in a data series before further parametric analysis can be conducted. The 4253HT provides a useful tool for analyzing trend without losing important features of a data series.

Many researchers in various areas including finance (Polasek, 1984); climatology (Reynolds, 1988); microbiology (Ashelford et al., 1999); agriculture (Mozny et al., 2009); seismology (Hird and McDermid, 2009); medical (Stern and Lightfoot, 1999) and (Verma et al., 2015) and image signal processing Dibeklioglu et al. (2017) opted 4253HT in preliminary analysis for trend trajectory. Some comparison studies have also been conducted to measure the effectiveness of the smoothers via simulation and practical analysis involving real data. The 4253HT manages to perform outstandingly as other existing non-linear smoothers.

However, 4253HT does not respond well to oscillated trend, (Tóthmérész and Erdei, 1995) and (Jin and Xu, 2013). The number of observations for 4253HT should be at least seven or else it would converge to constant root (Janosky et al., 1997). Velleman's compound smoother has also been revised by trying out possible combinations of multiple steps of running median, Hanning and re-smoothing of the rough (Sargent and Bedford, 2010). On the other hand, improvement on existing compound smoother has yet been explored in depth.

II. Methodology

Compound Smoother

In some cases, one type of smoother is not sufficient to reduce noise in a data series. Short span size of running median may apply non-resistance to the impulse and long span size may eliminate noise excessively, resulting in over-smoothing. Hence, (Tukey, 1977) introduced compound smoother that combines repeated running median of various spans, weighted moving average (Hanning), splitting and smoothing the rough.

Running Median

The standard median smoother in a window is obtained when odd numbers of successive observations in a sequence are arranged in ascending manner and the middle or median value is used as the output. Median is the best measure of tendency in a case where the data are disturbed by outliers. It is not affected by unrepresentative values in a data series. The median will provide the most probable value where all neighboring values are lie.

Let X_t be the input sequence with $t = 0, 1, \ldots, n$ and k is the span size with $k = 2u+1$ and $u = 1, 2, \ldots, n - \frac{k-1}{2}$. The general form of running median of odd span can be defined as

$$\tilde{X}_t = \text{median}(X_{t-\frac{k-1}{2}}, \ldots, X_{t-1}, X_t, X_{t+1}, \ldots, X_{t+\frac{k-1}{2}}). \quad (1)$$

Running median of odd span eliminates the number of spike or impulse according to span size. For example, span size 3 is capable to stabilize one sharp changes in a sequence. If there are two or more consecutive values that are falling apart from the series, running median of bigger span size is required (Justusson, 1981).

The median of even span is computed by averaging the two middle values in a sequence. Running median of even span is not an order statistic since the middle points are computed by using other function which is arithmetic mean.

The running median of k even span could be generally expressed by following the equation:

$$\tilde{X}_t = \text{median}\left(X_{t-\frac{k}{2}}, \ldots, X_t, \ldots, X_{t+\frac{k}{2}+1}\right) \quad (2)$$

where for even span size case $k = 2u$ with $u = 1, 2, \ldots, n - \frac{k}{2}$. Running median of odd span will follow the outlying pair; running median of even span size will be cut into half between two points with spike at the middle, by averaging them out.

Hanning

Hanning is another name for running weighted average. Running weighted average is important in making data cleaner and smooth. However, running weighted average is not resistant to outliers. Hence, Hanning is only applied after all outliers are removed by running median. (Tukey, 1977) uses a symmetric coefficients of form 1/4, 1/2, 1/4 of running weighted average as a gentle smoother after outliers are stabilized by a running median smoother, (Mills, 1991). The computation of Hanning is as follows :

$$h_t = \frac{1}{4}X_{t-1} + \frac{1}{2}X_t + \frac{1}{4}X_{t+1}. \quad (3)$$

Twice

The term "Twice" in compound smoother refers to the process of smoothing the rough. Re-smoothing the rough helps the recovery of the important features, eliminated due to repeated smoothing. For example, let $\{e_t\}$ denotes as rough which can be obtained by subtracting the actual values, $\{X_t\}$ and smoothed values computed from median smoother of span size three, $\{\tilde{X}_{t,3}\}$. The rough is re-smoothed by using median smoother of span size three and written as $\tilde{e}_{t,3} = \text{median}(e_{t-1}, e_t, e_{t+1})$. The re-smoothed rough is then added back to the smoothed values and expressed as the followings:

$$X_t = \tilde{X}_{t,3} + \tilde{e}_{t,3}. \quad (4)$$

4253HT

The 4253HT is a smoothing technique based on the combination of running median and running weighted averages. This technique was introduced by (Tukey, 1977) and described extensively by (Velleman, 1980). Let $\mathbf{X}$ be a doubly-infinite sequence of real data $\{X_{-n}, \ldots, X_{-1}, X_0, X_1, \ldots, X_n\}$. The computation of 4253HT starts with a running median of four, re-centered by running median of size 2.

$$S_1(\mathbf{X}_t) = \text{med}(X_{t-1}, X_t, X_{t+1}, X_{t+1}) \quad (5)$$

$$S_2(\mathbf{X}_t) = \text{med}[S_1(\mathbf{X}_t), S_1(\mathbf{X}_{t+1})] \quad (6)$$

Both Equation (5) and Equation (6) are running medians of even span size. The values are then re-smoothed by a running median of five and followed by running median of three.

$$\begin{aligned} S_3(\mathbf{X}_t) &= \text{med}[S_2(\mathbf{X}_{t-2}), S_2(\mathbf{X}_{t-1}), S_2(\mathbf{X}_t), \\ &\quad S_2(\mathbf{X}_{t+1}), S_2(\mathbf{X}_{t+2})] \end{aligned} \quad (7)$$

$$S_4(\mathbf{X}_t) = \text{med}\left[S_3(\mathbf{X}_{t-1}), S_3(\mathbf{X}_t), S_3(\mathbf{X}_{t+1})\right] \quad (8)$$

Subsequently, the values are computed using running weighted average of coefficients $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$(denoted as H for Hanning).

$$S_5(\mathbf{X}_t) = \frac{1}{4}S_4(\mathbf{X}_{t-1}) + \frac{1}{2}S_4(\mathbf{X}_t) + \frac{1}{4}S_4(\mathbf{X}_{t+1}) \quad (9)$$

The result of this smoothing is polished by computing the rough or residual and applying the same algorithm of smoothing. This process is called as "twicing". Then the result is added to the smoothed value obtained from the first pass.

$$S_6(\mathbf{X}_t) = S_5(\mathbf{X}_t) + S_5[\mathbf{X} - S_5(\mathbf{X}_t)] \quad (10)$$

Smoothing repeatedly could cause the lost of important features in a data series. "Twicing" helps in recovering Gaussian noise eliminated during smoothing process.

Modification of 4253HT

For even span, the middle point needs to be averaged out in order to obtain the smoothed value. This value is better than running median of odd span in the sense that it preserves significant spike in a data series. Middle point is computed by using arithmetic mean. The smoothed values after running median of size four and two are as follows

$$\mathbf{S}_2(\mathbf{X}_t) = \frac{1}{2}[\text{median}[(X_{t-2}, X_{t-1}, X_t, X_{t+1}) + \text{median}(X_{t-1}, X_t, X_{t+1}, X_{t+2})] \quad (11)$$

which X^* is the ordered observation from window in $X_{t-2}, X_{t-1}, X_t, X_{t+1}$ and X' is the ordered observation from window $X_{t-1}, X_t, X_{t+1}, X_{t+2}$. Some adjustments were proposed by applying different types of means. The types of means involved were geometric, quadratic, harmonic and contra harmonic. These types of means were chosen due to the relationship of contraharmonic $\geq$ quadratic $\geq$ arithmetic $\geq$ geometric $\geq$ harmonic. This inequality is only true for $X_t \geq 1$. If $X_t < 1$, this relationship will turn out otherwise.

The types of means that produce smaller value than arithmetic mean, namely geometric and harmonic, are expected to be more resistant to negative block pulse. On the other hand, quadratic and contra harmonic means are more responsive to positive changes in a data series.

Then, adjusted 4253HT using adaptive was developed by assigning each value to an algorithm that suits the specific condition of each value. Two instances of such condition include (i) negative impulse responds well to adjusted 4253HT using contra harmonic mean; and (ii) positive block pulse is resisted when adjusted 4253HT using harmonic mean is applied.

Hence, for running median of span size four and it being re-centered by span size two, the equation is as follows:

$$\mathbf{S}_2(X)_i = \begin{cases} \frac{X_{(i-1)}^2 + X_{(i)}^2 + X'^2_{(i)} + X'^2_{(i+1)}}{X_{(i-1)} + X_{(i)} + X'_{(i)} + X'_{(i+1)}} \text{ if } X_i \leq X_{i+1}, \\ \frac{1}{4}\left(\frac{1}{X_{(i-1)}} + \frac{1}{X_{(i)}} + \frac{1}{X'_{(i)}} + \frac{1}{X'_{(i+1)}}\right) \text{ if } X_i > X_{i+1} \end{cases} \quad (12)$$

However, the modifications would not work if the observation consists of zero or negative value. For example, geometric mean will generate the average as zero if there is any zero value in the computation. In addition, the value with negative sign will affect the true value of mean significantly. Hence, a constant point should be added to a data until all values are nonzero to ensure the smoothed value can be computed. Modification of 4253HT only involved the running median of span size 42 since it has a properties of preserving important edges without diminish excessively the main features of data series. The rough part was smoothed by applying the original algorithm where the middle points for running median of span size four and two were computed by arithmetic mean.

Simulation Procedure

The procedure of simulation conducted is according to (Conradie et al., 2009). In general, data can be described as follows:

$$\begin{aligned} \text{Data}_t &= \text{Signal}_t + \text{Noise}_t \\ &= \mu_t + D_t = X_t \end{aligned} \quad (13)$$

In this paper, four different types of signals were generated with noise with high volatility and heavy tailed distribution. Let the signals be special functions namely Doppler, HeaviSine, Bumps and Block. The formula for each function is as indicated by (Donoho and Johnstone, 1994). All formula are as follows:

Doppler

$$\mu_t = [t(1-t)]^{\frac{1}{2}} sin[2\pi(1+\epsilon)/(t+\epsilon)], \qquad \epsilon = 0.05 \quad (14)$$

Block

$$\begin{aligned} \mu_t &= \sum h_j K(t - t_j) \quad (15) \\ K(t) &= [1 + sgn(t)]/2 \\ t_j &= (0.1, 0.13, 0.15, 0.23, 0.25, 0.40, 0.44, \\ &\quad 0.65, 0.76, 0.78, 0.81) \\ (h_j) &= (4, -5, 3, -4, 5, -4.2, 2.1, 4.3, \\ &\quad -3.1, 2.1, -4.2) \end{aligned}$$

Bumps

$$\begin{aligned} \mu_t &= \sum h_j K(t - t_j/w_j), \quad (16) \\ K(t) &= (1 + |t|)^4 \\ t_j &= (0.1, 0.13, 0.15, 0.23, 0.25, 0.40, 0.44, \\ &\quad 0.65, 0.76, 0.78, 0.81) \\ (w_j) &= (0.005, 0.005, 0.006, 0.01, 0.01, 0.03, \\ &\quad 0.01, 0.05, 0.008, 0.005) \end{aligned}$$

HeaviSine

$$\mu_t = 4 \sin 4\pi t - \text{sgn}(t - 0.3) - \text{sgn}(0.72 - t) \quad (17)$$

Figure 1 illustrates the Doppler, Blocks, Bumps and HeaviSine signals.

The noise, $\{D_t\}$ was generated as i.i.d random variables from contaminated normal distribution. Let Z_t be a standard normal random variable and define the random variable D_t as:

$$D_t = \begin{cases} \alpha Z_t \text{ if } & Y_t = 1 \\ \beta Z_t \text{ if } & Y_t = 0 \end{cases} \quad (18)$$

with Y_t a Bernoulli(p) random variable independent from Z_t.

Thus $P(Y = 1) = p$ and $P(Y = 0) = 1 - p$ so that

$$\begin{aligned} P(D_t \leq d) &= P(\alpha Z_t \leq |Y_t = 1)P(Y_t = 1) + \\ &\quad P(\beta Z_t \leq d|Y_t = 0)P(Y_t = 0) \\ &= p\Phi\left(\frac{d}{\alpha}\right) + (1-p)\Phi\left(\frac{d}{\beta}\right) \quad (19) \end{aligned}$$

with $\{Z_t\}$ i.i.d $N(0, 1)$ and $\{Y_t\}$ i.i.d Bernoulli(p) and independent of the $\{Z_t\}$. In order to obtain a contaminated normal distribution with high kurtosis, the value of α is set as 5.06 so that for $p = 0.1$, $E(X^4) = 199.36$, $\text{Var}(X) = (0.1)(5.06)^2 + 0.9 = 3.46$ and kurtosis $= \frac{199.36}{(3.46)^2} = 16.65$. In the simulation of the contaminated normal distribution, approximately 10% of the values come from a $N(0, 5.06^2)$ distribution and approximately 90% from a $N(0, 1)$ distribution.

However, the p value at 10% does not affect the variation of the signal significantly. Therefore, value of p=0.75 was used for the purpose of comparing the performance of 4253HT in capturing the signal in almost unrecognized heavy noise. By increasing the percentage of contaminated error, the performance of 4253HT in extracting signal from heavy noises are measurable in a more meaningful manner.

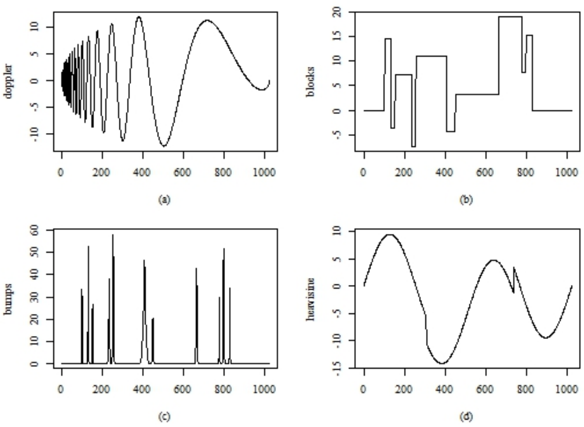

Figure 1: Signal of (a) Doppler (b) Block (c) Bumps and (d) HeaviSine

Performance Measurement

There were two approaches employed to measure the performance of the smoother in recovering the signal from heavy noise. The first one is by computing the regression coefficient, β_1 via the following equation:

$$X_t = \beta_0 + \beta_1 \mu_t + \epsilon_t \quad (20)$$

The value of β_1 measures the ratio of variation between signal and smoother over the variation of signal. If β_1 is close to one, the smoother is considered to be performing well in extracting signal from heavy noise.

Another approach is Estimated Integrated Mean Square Error (EIMSE) which was measured using the following equation:

$$\frac{1}{N}\sum_{t=1}^{N}\frac{1}{k}\sum_{j=1}^{k}(X_{jt}-\mu_j)^2 \tag{21}$$

EIMSE computes the average distance between signal and smoother. The lower the value of EIMSE, the better the performance of smoother in reducing unwanted noise in a series.

III. Results and Discussion

Tables 1 and 2 show the performance of the original and modified 4253HT in extracting signal from 10% and 75% contaminated normal noise measured by regression coefficient. Tables 3 and 4 depict the performance of the original and modified 4253HT in extracting signal from 10% and 75% contaminated normal noise measured by EIMSE. Let M1 represents the original 4253HT while M2, M3, M4, M5 and M6 represent modified 4253HT using geometric, quadratic, harmonic, contra harmonic and adaptive means respectively.

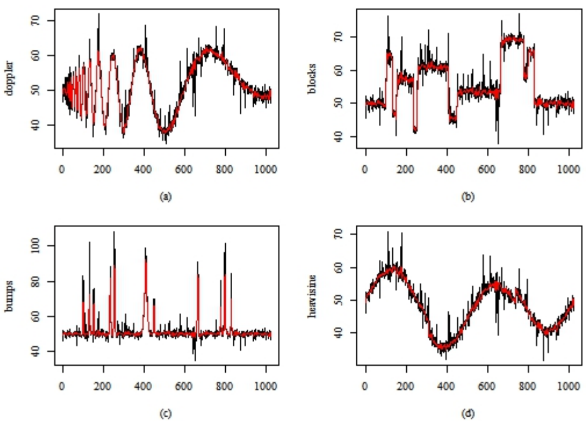

Figure 2: Performance of adjusted 4253HT in extracting signal with 10% contaminated normal noise

From the results of EIMSE, modification with adaptive mean was found to perform better than original, geometric, quadratic, and contraharmonic in reducing the noise of 10% and 75% contaminated normal from Doppler, Blocks, Bumps and Heavy Sine. On the other hand, modification with harmonic mean is the second best option among all.

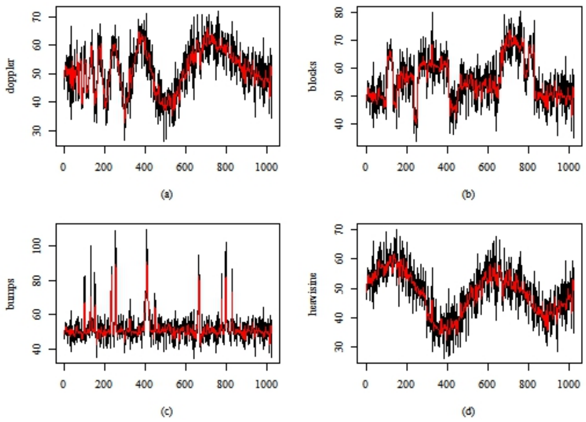

Figure 3: Performance of adjusted 4253HT in extracting signal with 75% contaminated normal noise

Table 1: The regression coefficient of original and modified 4253HT for signal added with 10% contaminated normal noise

Smoother	Signal	
	Doppler	Block
M1	0.942498	0.929867
M2	0.942509	0.929808
M3	0.942472	0.929844
M4	0.942544	0.929868
M5	0.942447	0.929817
M6	**0.942571**	**0.930159**
	Bumps	Heavy Sine
M1	0.802531	0.942553
M2	0.800696	0.942569
M3	0.804362	0.942542
M4	0.798964	0.942576
M5	**0.806256**	0.942530
M6	0.803996	**0.942602**

The results of regression coefficients show that modified 4253HT using adaptive mean performs the best in extracting the noise of 10% and 75% contaminated normal for all signals except Bumps. For bumps signal, the results

Table 2: The regression coefficient of original and modified 4253HT for signal added with 75% contaminated normal noise

Smoother	Signal	
	Doppler	Block
M1	0.775604	0.777160
M2	0.775814	0.777375
M3	0.775429	0.776981
M4	0.775993	0.777506
M5	0.775257	0.776805
M6	**0.776406**	**0.778050**
	Bumps	Heavy Sine
M1	0.679085	0.780078
M2	0.677523	0.780254
M3	0.680683	0.779903
M4	0.675979	0.780475
M5	**0.682346**	0.779718
M6	0.679644	**0.780808**

Table 3: The EIMSE of original and modified 4253HT for signal added with 10% contaminated normal noise

Smoother	Signal	
	Doppler	Block
M1	3.41195	3.67783
M2	3.41389	3.68123
M3	3.41178	3.67832
M4	3.41275	3.68239
M5	3.41171	3.67993
M6	**3.40788**	**3.66146**
	Bumps	Heavy Sine
M1	6.75217	3.27487
M2	6.781289	3.27702
M3	6.72926	3.27455
M4	6.80953	3.27577
M5	6.70822	3.27431
M6	**6.70801**	**3.27182**

of regression coefficients show that modified 4253HT using harmonic mean has the ability to reduce noise of 10% and 75% contaminated normal.

Table 4: The EIMSE of original and modified 4253HT for signal added with 75% contaminated normal noise

Smoother	Signal	
	Doppler	Block
M1	19.11529	19.26237
M2	19.13018	19.25171
M3	19.11074	19.25110
M4	19.13077	19.26205
M5	19.10865	19.25357
M6	**19.05288**	**19.17373**
	Bumps	Heavy Sine
M1	22.49567	18.95413
M2	22.53439	18.96762
M3	22.47254	18.94970
M4	22.56581	18.96833
M5	22.45481	18.94833
M6	**22.41773**	**18.89167**

Practical Application

From the results of simulation based on EIMSE, it is found that adaptive 4253HT works best in Bumps function, with 10% outliers and under the condition of high volatility. In this study, the daily amount of rainfall recorded at University of Malaya, Kuala Lumpur station in 2006 was applied since the features mimic the pattern of Bumps function.

Figure 4 shows the time series plot of the daily amount of rainfall at University of Malaya, Kuala Lumpur station in 2006 (in millimeter). The highest amount of rainfall recorded was 27 mm per day and on average, amount of rainfall in 2006 was 4.7 mm. Table III. shows the rain status based on the amount of rainfall retrieved from http://forecast.water.gov.my/. From Table III., station at University of Malaya, Kuala Lumpur never experienced heavy rain in 2006.

The daily amount of rainfall consists of some values of zero which indicates there was no rain. Since the adjusted adaptive 4253HT does not work under the condition of zero and negative values, a constant value of 10 was added to the data series and then subtracted back by 10 after

Table 5: Rainfall indicator in an hour

Category	Amount of rainfall (mm)
Light	< 10
Moderate	11-30
Heavy	30-60
Very Heavy	> 60

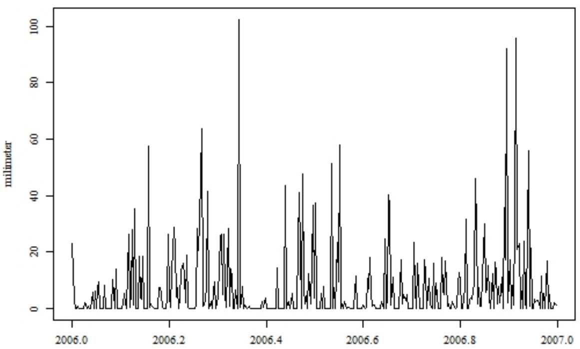

Figure 4: Daily amount of rainfall at University of Malaya, Kuala Lumpur station in 2006

the smoothed values were computed. As soon

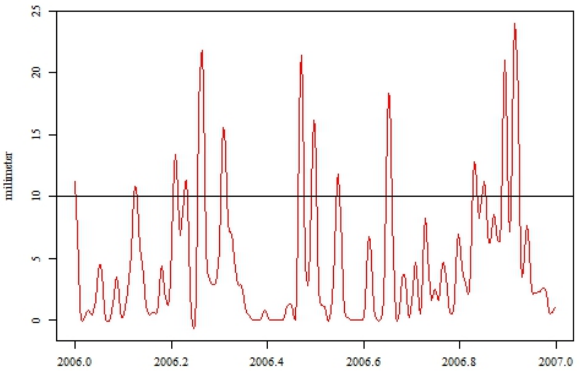

Figure 5: Plot of smoothed data using adjusted 4253HT

as the adjusted adaptive 4253HT smoother was applied to the data, the dispersion of daily amount rainfall became clearer. From Figure III., there was no regular fluctuation or seasonality observed in the data series. An interesting feature found was that the occurrences of moderate rain continued on for the next four days on average.

Conclusion

This study presents some modifications of existing compound smoother, 4253HT. The adjustment involved running median of span size four and two. Traditionally, median for even span size is computed by averaging the two middle points of arranged sequence arithmetically in a window. In some cases, different types of mean give better result in robustness and preserving edge. Therefore, five types of means were incorporated in the study, including geometric, quadratic, harmonic and contra harmonic. Adjusted 4253HT using adaptive was developed by assigning each value to an algorithm that suits the specific condition of each value. The performance was assessed via simulation study. Special functions of Doppler, Block, Bumps and Heavy Sine added with contaminated normal noise were generated. Adjusted adaptive 4253HT was found to perform best in extracting signal from noise with heavy-tailed distribution and high volatility. The elaborated application to the daily amount of rainfall provides clearer vision- that on average, heavy rain would occur for five consecutive days.

Acknowledgements

This research is partially funded from Putra Grant GP/2018/969400.

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